Problem 1. Consider the system $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}. $$

1. Project $b$ onto the column space of $A$ by solving $A^T A \tilde{x} = A^T b$.
2. Compute $p = A\tilde{x}$.
3. Find $e = b - p$ and show it is orthogonal to the columns of $A$.

Problem 2. Find two orthogonal vectors that lie in the plane $x + y + 2z = 0$. Make them orthonormal.

Problem 3.

1. Find orthonormal vectors $q_1$, $q_2$ that span the column space of

$$ \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}. $$

2. Compute $q_3$ such that $q_1$, $q_2$, $q_3$ are orthonormal.
3. Which of the four subspaces contains $q_3$?
4. Solve $Ax = (1, 2, 7)^T$ by least squares.

Problem 4.

1. What multiple of the vector $A^T = a^T = (1, 1)$ should be subtracted from the vector $b^T = (4, 0)$ to make the resulting vector $B$ orthogonal to $A$?
2. Complete the Gram-Schmidt process by computing $q_1 = A/\|A\|$ and $q_2 = B/\|B\|$.
3. Factor The matrix $Z = [A, B]$ into QR:

$$ Z = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \|A\| & ? \\ 0 & \|B\| \end{bmatrix}. $$
Problem 5.
1. What multiple of $a^T = (1, 1)$ should be subtracted from $b^T = (4, 0)$ to make the result $c$ orthogonal to $a$?
2. Complete the Gram-Schmidt process by computing $q_1 = a/\|a\|$ and $q_2 = c/\|c\|$.
3. Factor $A = [a, b]$ into QR:
   \[
   \begin{bmatrix}
   1 & 4 \\
   1 & 0
   \end{bmatrix} = \begin{bmatrix}
   q_1 \\
   q_2
   \end{bmatrix} \begin{bmatrix}
   \|a\| & ? \\
   0 & \|c\|
   \end{bmatrix}.
   \]

Problem 6. Determine the linear combination of the two vectors $a = (1, 2, -1)$ and $b = (1, 0, 1)$ which is closest to the vector $c = (-1, 1, 1)$. Explain your results.

Problem 7. Assume the parabola $b = C + Dt + Et^2$ with measurements $b = 0, 8, 8, 20$ at times $t = 0, 1, 3, 4$.
1. Write down the four equations $Ax = b$.
2. Set up and solve the normal equations $A^T A x = A^T b$.
3. Let $e_i = b_i - C - Dt_i$. Compute $e_1 + e_2 + e_3 + e_4$.
4. What is the minimum of $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?
5. Compute $p = A\bar{x}$. Verify that $e = b - p$ is orthogonal to both columns of $A$.
6. What is the shortest distance $\|e\|$ from $b$ to the column space?

Problem 8. Assume the straight line $b = C + Dt$ with measurements $b = 10, 4, 4, 0$ for $t_i = -1, 0, 1, 4$.
1. Write down the four equations $Ax = b$.
2. Set up and solve the normal equations $A^T A \bar{x} = A^T b$ for $\bar{x} = (C, D)$.
3. Compute $p = A\bar{x}$ and verify that $e = b - p$ is orthogonal to both columns of $A$.
4. Let $e_i = b_i - C - Dt_i$. Compute $e_1 + e_2 + e_3 + e_4$ and $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$.
5. Compute the shortest distance from $b$ to the column space of $A$. 

Problem 9. Project \( b = (b_1, \cdots, b_m) \) onto the line through \( a = (1, \cdots, 1) \).

1. Solve \( a^T a \bar{x} = a^T b \) to show that \( \bar{x} \) is the mean of the \( b \)'s.

2. Find the error vector \( e \), the variance \( \|e\|^2 \), and the standard deviation \( \|e\| \).

3. Draw a graph with \( b = (1, 2, 6) \) fitted to a horizontal line. What are \( p \) and \( e \) on the graph? Check that \( e \) is orthogonal to \( p \).

Problem 10. Are the following pairs of vectors independent, orthogonal, or orthonormal? Change the second vector when necessary to produce orthonormal vectors.

1. \( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \).

2. \( \begin{bmatrix} .6 \\ .8 \end{bmatrix}, \begin{bmatrix} .4 \\ -.3 \end{bmatrix} \).

3. \( \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \).

Problem 11. Find orthogonal vectors by Gram-Schmidt from

\[
\begin{align*}
a &= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, & b &= \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, & c &= \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.
\end{align*}
\]