Answer at least three of the next four problems

Problem I. Consider the following rank one matrix:

\[ A = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}. \]

(Hint: It is not necessary to compute \( A \) to do this problem.)

1. Find the four fundamental spaces of \( A \).
2. Find the eigenvalues of \( A \).
3. Find the eigenvectors of \( A \).

Problem II. Consider the matrix:

\[ A = \begin{bmatrix} 1 & 2 & 3 & 6 \end{bmatrix}. \]

1. Compute \( A^T A \) and its eigenvalues and unit eigenvectors \( v_1 \) and \( v_2 \).
2. Show that the rank of \( A \) is 1.
3. What is, \( \sigma_1 \), the nonzero singular value of \( A \)?
4. Compute \( AA^T \) and its eigenvalues and unit eigenvectors \( u_1 \) and \( u_2 \).
5. Verify that \( Av_1 = \sigma_1 u_1 \).
6. Write down the Singular Value Decomposition of \( A \).
7. Write down orthonormal bases for the four fundamental subspaces for the matrix \( A \).

Problem III. What values of \( b \) makes this matrix positive definite?

\[ A = \begin{bmatrix} 1 \\ b \\ b \end{bmatrix} \begin{bmatrix} 1 & b & 9 \end{bmatrix}. \]
Problem IV.

1. Compute the eigenvalues and eigenvector of $A$, $B$, $A + B$, $AB$, and $BA$:

\[
A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},
\]

2. Are the eigenvalues of $A + B$ equal to the eigenvalues of $A$ plus the eigenvalues of $B$?

3. Are the eigenvalues of $AB$ equal to the eigenvalues of $A$ times the eigenvalues of $B$?

4. Are the eigenvalues of $BA$ equal to the eigenvalues of $B$ times the eigenvalues of $A$?

Answer at least one of the next two problems

Problem V. Consider the block matrix $B = \begin{bmatrix} A & D \\ D^T & C \end{bmatrix}$. Assume that $A$ and $C$ are symmetric. Also assume $B$ satisfies $x^T B x > 0$ for all vectors $x \neq 0$ (positive definite).

1. Show that $B$ is symmetric and $(A^{-1})^T = A^{-1}$.

2. Let $x = \begin{bmatrix} y \\ z \end{bmatrix}$. Compute

\[
x^T B x = \begin{bmatrix} y^T, z^T \end{bmatrix} \begin{bmatrix} A & D \\ D^T & C \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}
\]

3. Show that $A$ and $C$ are positive definite.

4. Show that the matrix $S = C - D^T A^{-1} D$ is also positive definite.

Problem VI.

1. Show that the block matrix $B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$, is symmetric even when $A$ is rectangular.

2. Show that if $B x = \lambda x$ then $A z = \lambda y$ and $A^T y = \lambda z$, where $x = \begin{bmatrix} y \\ z \end{bmatrix}$.

3. Show that $A^T A z = \lambda^2 z$ so $\lambda^2$ is an eigenvalue of $A^T A$.

4. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find all three eigenvalues and eigenvectors of $B$.

Extra credit will be given if you answer both Problems V and VI correctly. Please state which one is for extra credit.