6. (i) By (5.28) with $L = 1$ and $c = 1$,

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\big[b_{1n}\cos(n\pi t) + b_{2n}\sin(n\pi t)\big]$$

$$\Rightarrow \quad u(x,0) = -3\sin(2\pi x) + 4\sin(7\pi x) = \sum_{n=1}^{\infty} b_{1n}\sin(n\pi x)$$

$$\Rightarrow \quad b_{12} = -3, \quad b_{17} = 4, \quad b_{1n} = 0,\ n \neq 2, 7,$$

$$u_t(x,t) = \sum_{n=1}^{\infty} n\pi\sin(n\pi x)\big[-b_{1n}\sin(n\pi t) + b_{2n}\cos(n\pi t)\big]$$

$$\Rightarrow \quad u_t(x,0) = \sin(3\pi x) = \sum_{n=1}^{\infty} n\pi b_{2n}\sin(n\pi x)$$

$$\Rightarrow \quad 3\pi b_{23} = 1, \quad n\pi b_{2n} = 0,\ n \neq 3$$

$$\Rightarrow \quad u(x,t) = -3\sin(2\pi x)\cos(2\pi t) + 4\sin(7\pi x)\cos(7\pi t) + \frac{1}{3\pi}\sin(3\pi x)\sin(3\pi t).$$

(ii) Here

$$b_{1n} = 2\int_0^1 f(x)\sin(n\pi x)\,dx = -2\int_0^1 \sin(n\pi x)\,dx$$

$$= \frac{2}{n\pi}\big[\cos(n\pi x)\big]_0^1 = \frac{2}{n\pi}\big[\cos(n\pi) - 1\big] = \big[(-1)^n - 1\big]\frac{2}{n\pi}, \quad n = 1, 2, \ldots$$

$$\pi b_{21} = 3, \quad n\pi b_{2n} = 0,\ n \neq 1$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty}\big[(-1)^n - 1\big]\frac{2}{n\pi}\sin(n\pi x)\cos(n\pi t) + \frac{3}{\pi}\sin(\pi x)\sin(\pi t).$$

(iii) Using the ICs (with $L = 1$), we have

$$b_{13} = 2, \quad b_{1n} = 0,\ n \neq 3,$$

$$b_{2n} = \frac{2}{n\pi}\int_0^1 g(x)\sin(n\pi x)\,dx = \frac{2}{n\pi}\int_0^1 2\sin(n\pi x)\,dx$$

$$= -\frac{4}{n^2\pi^2}\big[\cos(n\pi x)\big]_0^1 = \big[1 - (-1)^n\big]\frac{4}{n^2\pi^2}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = 2\sin(3\pi x)\cos(3\pi t) + \sum_{n=1}^{\infty} \big[1-(-1)^n\big]\,\frac{4}{n^2\pi^2}\,\sin(n\pi x)\sin(n\pi t).$$

(iv) With $L = 1$ and $c = 1$,

$$\begin{aligned}
b_{1n} &= 2\int_0^1 \sin(n\pi x)\,dx = -\frac{2}{n\pi}\,\big[\cos(n\pi x)\big]_0^1 = -\frac{2}{n\pi}\,\big[\cos(n\pi) - 1\big] \\
&= \big[1-(-1)^n\big]\,\frac{2}{n\pi}, \quad n = 1, 2, \ldots, \\
b_{2n} &= \frac{2}{n\pi}\int_0^1 x\sin(n\pi x)\,dx \\
&= \frac{2}{n\pi}\left\{\left[x\left(-\frac{1}{n\pi}\right)\cos(n\pi x)\right]_0^1 + \int_0^1 \frac{1}{n\pi}\,\cos(n\pi x)\,dx\right\} \\
&= \frac{2}{n\pi}\left\{-\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n^2\pi^2}\,\big[\sin(n\pi x)\big]_0^1\right\} \\
&= (-1)^{n+1}\,\frac{2}{n^2\pi^2}, \quad n = 1, 2, \ldots
\end{aligned}$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left\{\big[1-(-1)^n\big]\,\frac{2}{n\pi}\,\cos(n\pi t) + (-1)^{n+1}\,\frac{2}{n^2\pi^2}\,\sin(n\pi t)\right\}.$$

7. (i) By (5.30) with $L = 1$ and $c = 1$,

$$\begin{aligned}
& u(x,t) = a_{10} + a_{20}t + \sum_{n=1}^{\infty} \cos(n\pi x)\big[a_{1n}\cos(n\pi t) + a_{2n}\sin(n\pi t)\big] \\
\Rightarrow\quad & u(x,0) = 2 - 3\cos(4\pi x) = a_{10} + \sum_{n=1}^{\infty} a_{1n}\cos(n\pi x) \\
\Rightarrow\quad & a_{10} = 2, \quad a_{14} = -3, \quad a_{1n} = 0,\ n \neq 0, 4, \\
& u_t(x,t) = a_{20} + \sum_{n=1}^{\infty} n\pi\cos(n\pi x)\big[-a_{1n}\sin(n\pi t) + a_{2n}\cos(n\pi t)\big] \\
\Rightarrow\quad & u_t(x,0) = 2\cos(3\pi x) = a_{20} + \sum_{n=1}^{\infty} n\pi a_{2n}\cos(n\pi x) \\
\Rightarrow\quad & 3\pi a_{23} = 2, \quad n\pi a_{2n} = 0,\ n \neq 3
\end{aligned}$$

$$\Rightarrow \quad u(x,t) = 2 - 3\cos(4\pi x)\cos(4\pi t) + \frac{2}{3\pi}\,\cos(3\pi x)\sin(3\pi t).$$

(ii) With $L = 1$, from the ICs we have

$$\begin{aligned}
a_{10} &= \int_0^1 f(x)\,dx = \int_0^1 (x-1)\,dx = \left[\tfrac{1}{2}\,x^2 - x\right]_0^1 = -\tfrac{1}{2}, \\
a_{1n} &= 2\int_0^1 f(x)\cos(n\pi x)\,dx = 2\int_0^1 (x-1)\cos(n\pi x)\,dx \\
&= 2\left\{\left[(x-1)\,\frac{1}{n\pi}\,\sin(n\pi x)\right]_0^1 - \int_0^1 \frac{1}{n\pi}\,\sin(n\pi x)\,dx\right\} \\
&= \frac{2}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^1 = \left[(-1)^n - 1\right]\frac{2}{n^2\pi^2}, \quad n = 1, 2, \ldots, \\
a_{20} &= 2, \quad \pi a_{21} = -1, \quad a_{2n} = 0,\ n \neq 0, 1
\end{aligned}$$

$$\Rightarrow \; u(x,t) = -\frac{1}{2} + 2t + \sum_{n=1}^{\infty}\left[(-1)^n - 1\right]\frac{2}{n^2\pi^2}\,\cos(n\pi x)\cos(n\pi t) - \frac{1}{\pi}\,\cos(\pi x)\sin(\pi t).$$

(iii) Here

$$\begin{aligned}
a_{12} &= -3, \quad a_{1n} = 0,\ n \neq 2, \\
a_{20} &= \int_0^1 g(x)\,dx = \int_0^1 (2x-1)\,dx = \left[x^2 - x\right]_0^1 = 0, \\
a_{2n} &= \frac{2}{n\pi}\int_0^1 g(x)\cos(n\pi x)\,dx = \frac{2}{n\pi}\int_0^1 (2x-1)\,\cos(n\pi x)\,dx \\
&= \frac{2}{n\pi}\left\{\left[(2x-1)\,\frac{1}{n\pi}\,\sin(n\pi x)\right]_0^1 - \int_0^1 \frac{1}{n\pi}\,\sin(n\pi x)\cdot 2\,dx\right\} \\
&= \frac{2}{n\pi}\,\frac{2}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^1 = \left[(-1)^n - 1\right]\frac{4}{n^3\pi^3}, \quad n = 1, 2, \ldots
\end{aligned}$$

$$\Rightarrow \quad u(x,t) = -3\cos(2\pi x)\cos(2\pi t) + \sum_{n=1}^{\infty}\left[(-1)^n - 1\right]\frac{4}{n^3\pi^3}\,\cos(n\pi x)\sin(n\pi t).$$

(iv) By (5.31) with $L = 1$ and $c = 1$,

$$a_{10} = \int_0^1 x\,dx = \tfrac{1}{2}\left[x^2\right]_0^1 = \tfrac{1}{2},$$

$$a_{1n} = 2\int_0^1 x\cos(n\pi x)\,dx = 2\left\{\left[x\,\frac{1}{n\pi}\,\sin(n\pi x)\right]_0^1 - \int_0^1 \frac{1}{n\pi}\,\sin(n\pi x)\,dx\right\}$$

$$= \frac{2}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^1 = \left[(-1)^n - 1\right]\frac{2}{n^2\pi^2}, \quad n = 1, 2, \ldots,$$

$$a_{20} = \int_0^1 (2x-1)\,dx = \left[x^2 - x\right]_0^1 = 0,$$

$$a_{2n} = \frac{2}{n\pi}\int_0^1 (2x-1)\cos(n\pi x)\,dx$$

$$= \frac{2}{n\pi}\left\{\left[(2x-1)\,\frac{1}{n\pi}\,\sin(n\pi x)\right]_0^1 - \int_0^1 \frac{1}{n\pi}\,\sin(n\pi x)\cdot 2\,dx\right\}$$

$$= \frac{2}{n\pi}\,\frac{2}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^1 = \left[(-1)^n - 1\right]\frac{4}{n^3\pi^3}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \cos(n\pi x)\left\{\frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]\cos(n\pi t) + \left[(-1)^n - 1\right]\frac{4}{n^3\pi^3}\,\sin(n\pi t)\right\}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left[(-1)^n - 1\right]\frac{2}{n^2\pi^2}\,\cos(n\pi x)\left[\cos(n\pi t) + \frac{2}{n\pi}\,\sin(n\pi t)\right].$$