3. (i) As in Example 5.5,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{(2n-1)\pi x}{2}\, e^{-(2n-1)^2\pi^2 t/4}$$

$$\Rightarrow \quad u(x,0) = -3 = \sum_{n=1}^{\infty} c_n \sin\frac{(2n-1)\pi x}{2},$$

$$c_n = 2\int_0^1 -3\sin\frac{(2n-1)\pi x}{2}\,dx = (-6)\left(-\frac{2}{(2n-1)\pi}\right)\left[\cos\frac{(2n-1)\pi x}{2}\right]_0^1$$

$$= -\frac{12}{(2n-1)\pi}, \qquad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = -\sum_{n=1}^{\infty} \frac{12}{(2n-1)\pi}\, \sin\frac{(2n-1)\pi x}{2}\, e^{-(2n-1)^2\pi^2 t/4}.$$

(ii) Similarly,

$$u(x,0) = 2 + x = \sum_{n=1}^{\infty} c_n \sin\frac{(2n-1)\pi x}{2},$$

$$c_n = 2\int_0^1 (2+x)\sin\frac{(2n-1)\pi x}{2}\,dx$$

$$= 2\left\{\left[(2+x)\left(-\frac{2}{(2n-1)\pi}\right)\cos\frac{(2n-1)\pi x}{2}\right]_0^1 + \int_0^1 \frac{2}{(2n-1)\pi}\,\cos\frac{(2n-1)\pi x}{2}\,dx\right\}$$

$$= 2\left\{\frac{4}{(2n-1)\pi} + \frac{4}{(2n-1)^2\pi^2}\left[\sin\frac{(2n-1)\pi x}{2}\right]_0^1\right\}$$

$$= \frac{8}{(2n-1)\pi} + \frac{8}{(2n-1)^2\pi^2}\,\sin\frac{(2n-1)\pi}{2}$$

$$= \frac{8}{(2n-1)\pi} + (-1)^{n+1}\frac{8}{(2n-1)^2\pi^2}, \qquad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty}\left[\frac{8}{(2n-1)\pi} + (-1)^{n+1}\frac{8}{(2n-1)^2\pi^2}\right]\sin\frac{(2n-1)\pi x}{2}\, e^{-(2n-1)^2\pi^2 t/4}.$$

4. (i) As in Example 5.6,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2}\, e^{-(2n-1)^2\pi^2 t/4}$$

$$\Rightarrow \quad u(x,0) = 4 = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2},$$

$$c_n = 2\int_0^1 4\cos\frac{(2n-1)\pi x}{2}\, dx = 8\,\frac{2}{(2n-1)\pi}\left[\sin\frac{(2n-1)\pi x}{2}\right]_0^1$$

$$= \frac{16}{(2n-1)\pi}\,\sin\frac{(2n-1)\pi}{2} = (-1)^{n+1}\,\frac{16}{(2n-1)\pi}, \qquad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{16}{(2n-1)\pi}\,\cos\frac{(2n-1)\pi x}{2}\, e^{-(2n-1)^2\pi^2 t/4}.$$

(ii) Here

$$u(x,0) = 2x - 3 = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2},$$

$$c_n = 2\int_0^1 (2x-3)\cos\frac{(2n-1)\pi x}{2}\, dx$$

$$= 2\left\{\left[(2x-3)\,\frac{2}{(2n-1)\pi}\,\sin\frac{(2n-1)\pi x}{2}\right]_0^1 - \int_0^1 \frac{2}{(2n-1)\pi}\,\sin\frac{(2n-1)\pi x}{2}\cdot 2\, dx\right\}$$

$$= 2\left\{-\frac{2}{(2n-1)\pi}\,\sin\frac{(2n-1)\pi}{2} + \frac{8}{(2n-1)^2\pi^2}\left[\cos\frac{(2n-1)\pi x}{2}\right]_0^1\right\}$$

$$= (-1)^n\,\frac{4}{(2n-1)\pi} - \frac{16}{(2n-1)^2\pi^2}, \qquad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty}\left[(-1)^n\,\frac{4}{(2n-1)\pi} - \frac{16}{(2n-1)^2\pi^2}\right]\cos\frac{(2n-1)\pi x}{2}\, e^{-(2n-1)^2\pi^2 t/4}.$$

5. (i) As in Example 5.7,

$$
\begin{aligned}
& u(x,t) = a_0 + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x)\big] e^{-n^2\pi^2 t} \\
\Rightarrow\quad & u(x,0) = 2\sin(2\pi x) - \cos(5\pi x) = a_0 + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x)\big] \\
\Rightarrow\quad & a_5 = -1, \quad a_n = 0,\ n \neq 5, \quad b_2 = 2, \quad b_n = 0,\ n \neq 2 \\
\Rightarrow\quad & u(x,t) = 2\sin(2\pi x) e^{-4\pi^2 t} - \cos(5\pi x) e^{-25\pi^2 t}.
\end{aligned}
$$

(ii) By (5.23) and (5.24) with $L = 1$ and $k = 1$,

$$
\begin{aligned}
& u(x,t) = a_0 + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x)\big] e^{-n^2\pi^2 t} \\
\Rightarrow\quad & u(x,0) = 3x - 2 = a_0 + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x)\big], \\
& a_0 = \frac{1}{2} \int_{-1}^{1} (3x - 2)\, dx = \tfrac{1}{2} \big[\tfrac{3}{2}\, x^2 - 2x\big]_{-1}^{1} = -2, \\
& a_n = \int_{-1}^{1} (3x - 2) \cos(n\pi x)\, dx \\
& \quad = \left[(3x - 2)\, \frac{1}{n\pi}\, \sin(n\pi x)\right]_{-1}^{1} - \int_{-1}^{1} \frac{1}{n\pi}\, \sin(n\pi x) \cdot 3\, dx \\
& \quad = \frac{3}{n^2\pi^2} \big[\cos(n\pi x)\big]_{-1}^{1} = 0, \quad n = 1, 2, \ldots, \\
& b_n = \int_{-1}^{1} (3x - 2) \sin(n\pi x)\, dx \\
& \quad = \left[(3x - 2)\left(-\frac{1}{n\pi}\right) \cos(n\pi x)\right]_{-1}^{1} + \int_{-1}^{1} \frac{1}{n\pi}\, \cos(n\pi x) \cdot 3\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{n\pi}\,\cos(n\pi) - \frac{5}{n\pi}\,\cos(n\pi) + \frac{3}{n^2\pi^2}\,\big[\sin(n\pi x)\big]_{-1}^{1} \\
&= (-1)^{n+1}\frac{6}{n\pi}, \quad n = 1, 2, \dots
\end{aligned}
$$

$$
\Rightarrow \quad u(x,t) = -2 + \sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{6}{n\pi}\,\sin(n\pi x)\,e^{-n^2\pi^2 t}.
$$