Chapter 5

1. (i) Proceeding as in Example 5.2, we have

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}$$

$$\Rightarrow \quad u(x,0) = \sin(2\pi x) - 3\sin(6\pi x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$\Rightarrow \quad b_2 = 1, \quad b_6 = -3, \quad b_n = 0,\ n \neq 2, 6$$

$$\Rightarrow \quad u(x,t) = \sin(2\pi x) e^{-4\pi^2 t} - 3\sin(6\pi x) e^{-36\pi^2 t}.$$

The coefficients b_n can also be obtained by means of formulas (5.9).

(ii) Here, by (5.9) and (5.10) with $L = 1$ and $k = 1$,

$$b_n = 2\int_0^1 -2\sin(n\pi x)\,dx = \frac{4}{n\pi}\big[\cos(n\pi x)\big]_0^1 = \frac{4}{n\pi}\big[\cos(n\pi) - 1\big]$$

$$= \big[(-1)^n - 1\big]\frac{4}{n\pi}, \quad n = 1, 2, \dots$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \big[(-1)^n - 1\big]\frac{4}{n\pi} \sin(n\pi x) e^{-n^2\pi^2 t}.$$

(iii) As in (ii),

$$b_n = 2\int_0^1 (2x+1)\sin(n\pi x)\,dx$$

$$= 2\bigg\{\bigg[(2x+1)\Big(-\frac{1}{n\pi}\Big)\cos(n\pi x)\bigg]_0^1 + \int_0^1 \frac{1}{n\pi}\cos(n\pi x)\cdot 2\,dx\bigg\}$$

$$= 2\bigg\{-\frac{1}{n\pi}\big[3\cos(n\pi) - 1\big] + \frac{2}{n^2\pi^2}\big[\sin(n\pi x)\big]_0^1\bigg\}$$

$$= \big[1 - 3(-1)^n\big]\frac{2}{n\pi}, \quad n = 1, 2, \dots$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \left[1 - (-1)^n 3\right] \frac{2}{n\pi} \sin(n\pi x) e^{-n^2\pi^2 t}.$$

(iv) Similarly,

$$\begin{aligned}
b_n &= 2 \int_0^{1/2} x \sin(n\pi x)\, dx \\
&= 2\left\{ \left[x\left(-\frac{1}{n\pi}\right) \cos(n\pi x) \right]_0^{1/2} + \int_0^{1/2} \frac{1}{n\pi} \cos(n\pi x)\, dx \right\} \\
&= 2\left\{ -\frac{1}{2n\pi} \cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2} \left[\sin(n\pi x)\right]_0^{1/2} \right\} \\
&= 2\left(-\frac{1}{2n\pi} \cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2} \sin\frac{n\pi}{2} \right), \quad n = 1, 2, \dots
\end{aligned}$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{n^2\pi^2} \sin\frac{n\pi}{2} - \frac{1}{n\pi} \cos\frac{n\pi}{2} \right) \sin(n\pi x) e^{-n^2\pi^2 t}.$$

2. (i) With the eigenvalues and eigenfunctions as in Example 5.4, we have

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-n^2\pi^2 t}$$

$$\begin{aligned}
\Rightarrow \quad & u(x,0) = 3 - 2\cos(4\pi x) = \sum_{n=1}^{\infty} a_n \cos(n\pi x) \\
\Rightarrow \quad & a_0 = 3, \quad a_4 = -2, \quad a_n = 0,\ n \neq 0, 4 \\
\Rightarrow \quad & u(x,t) = 3 - 2\cos(4\pi x) e^{-16\pi^2 t}.
\end{aligned}$$

(ii) By (5.14) with $L = 1$,

$$a_0 = \int_0^1 (2 - 3x)\, dx = \left[2x - \tfrac{3}{2}\, x^2\right]_0^1 = \tfrac{1}{2},$$

$$a_n = 2 \int_0^1 (2 - 3x) \cos(n\pi x)\, dx$$

$$= 2\left\{\left[(2-3x)\frac{1}{n\pi}\,\sin(n\pi x)\right]_0^1 - \int_0^1 \frac{1}{n\pi}\,\sin(n\pi x)\cdot(-3)\,dx\right\}$$

$$= 2\left\{-\frac{3}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^1\right\} = \frac{6}{n^2\pi^2}\left[1-\cos(n\pi)\right]$$

$$= \left[1-(-1)^n\right]\frac{6}{n^2\pi^2}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[1-(-1)^n\right]\frac{6}{n^2\pi^2}\,\cos(n\pi x)e^{-n^2\pi^2 t}.$$

(iii) As in (ii),

$$a_0 = \int_0^1 x^2\,dx = \tfrac{1}{3}\left[x^3\right]_0^1 = \tfrac{1}{3},$$

$$a_n = 2\int_0^1 x^2\cos(n\pi x)\,dx$$

$$= 2\left\{\left[x^2\frac{1}{n\pi}\,\sin(n\pi x)\right]_0^1 - \int_0^1 \frac{1}{n\pi}\,\sin(n\pi x)\cdot 2x\,dx\right\}$$

$$= \frac{4}{n\pi}\left\{\left[x\,\frac{1}{n\pi}\,\cos(n\pi x)\right]_0^1 - \int_0^1 \frac{1}{n\pi}\,\cos(n\pi x)\,dx\right\}$$

$$= \frac{4}{n\pi}\left\{\frac{1}{n\pi}\,\cos(n\pi) - \frac{1}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^1\right\} = (-1)^n\,\frac{4}{n^2\pi^2}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \frac{1}{3} + \sum_{n=1}^{\infty}(-1)^n\,\frac{4}{n^2\pi^2}\,\cos(n\pi x)e^{-n^2\pi^2 t}.$$

(iv) Here

$$a_0 = \int_{1/2}^1 2x\,dx = \left[x^2\right]_{1/2}^1 = \tfrac{3}{4},$$

$$
\begin{aligned}
a_n &= 2\int_{1/2}^{1} 2x\cos(n\pi x)\,dx = 4\left\{\left[x\,\frac{1}{n\pi}\,\sin(n\pi x)\right]_{1/2}^{1} + \int_{1/2}^{1}\frac{1}{n\pi}\,\sin(n\pi x)\,dx\right\}\\
&= 4\left\{-\frac{1}{2n\pi}\,\sin\frac{n\pi}{2} + \frac{1}{n^2\pi^2}\left[\cos(n\pi x)\right]_{1/2}^{1}\right\}\\
&= -\frac{2}{n\pi}\,\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]
\end{aligned}
$$

$$
\Rightarrow \quad u(x,t) = \frac{3}{4} + \sum_{n=1}^{\infty}\left\{-\frac{2}{n\pi}\,\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\right\}\cos(n\pi x)e^{-n^2\pi^2 t}.
$$