1. (iii) We have

$$2f'' + f' + \lambda f = 0 \quad \Leftrightarrow \quad e^{x/2} f'' + \tfrac{1}{2}\, e^{x/2} f' + \tfrac{1}{2}\, e^{x/2} \lambda f = 0$$
$$\Leftrightarrow \quad (e^{x/2} f')' + \tfrac{1}{2}\, \lambda e^{x/2} f = 0,$$

this is a singular S-L problem with

$$p(x) = e^{x/2}, \quad q(x) \equiv 0, \quad \sigma(x) = \tfrac{1}{2}\, e^{x/2},$$
$$a = 0, \quad b = \infty, \quad \kappa_1 = 1, \quad \kappa_2 = 0.$$

(iv) Here

$$x^2 f'' + x f' + \lambda(2 - x) f = 0 \quad \Leftrightarrow \quad x f'' + f' + \lambda\, \frac{2 - x}{x}\, f = 0$$
$$\Leftrightarrow \quad (x f')' + \lambda\, \frac{2 - x}{x}\, f = 0 \quad \text{(since } x \neq 0\text{)},$$

so this is a singular S-L problem with

$$p(x) = x, \quad q(x) \equiv 0, \quad \sigma(x) = \frac{2 - x}{x},$$
$$a = 0, \quad b = 1, \quad \kappa_3 = 1, \quad \kappa_4 = -1.$$

6. Using the standard procedure, we have

$$u(x) \sim a_0 + \sum_{n=1}^{\infty} \big[a_n \cos(nx) + b_n \sin(nx)\big],$$
$$\int_{-\pi}^{\pi} (2x + 3)\, dx = a_0 \int_{-\pi}^{\pi} dx = 2\pi a_0,$$
$$\Rightarrow \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (2x + 3)\, dx = \frac{1}{2\pi} \big[x^2 + 3x\big]_{-\pi}^{\pi} = 3,$$
$$\int_{-\pi}^{\pi} (2x + 3) \cos(mx)\, dx = a_m \int_{-\pi}^{\pi} \cos^2(mx)\, dx$$
$$= a_m \int_{-\pi}^{\pi} \tfrac{1}{2} \big[1 + \cos(2mx)\big]\, dx = \pi a_m$$

$$\Rightarrow \quad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} (2x+3)\cos(mx)\,dx$$

$$= \frac{1}{\pi}\left\{\left[(2x+3)\frac{1}{m}\,\sin(mx)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2}{m}\,\sin(mx)\,dx\right\}$$

$$= \frac{1}{\pi}\cdot\frac{2}{m^2}\,\big[\cos(mx)\big]_{-\pi}^{\pi} = 0,$$

$$\int_{-\pi}^{\pi} (2x+3)\sin(mx)\,dx = b_m\int_{-\pi}^{\pi} \sin^2(mx)\,dx$$

$$= b_m\int_{-\pi}^{\pi} \tfrac{1}{2}\big[1-\cos(2mx)\big]\,dx = \pi b_m$$

$$\Rightarrow \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} (2x+3)\sin(mx)\,dx$$

$$= \frac{1}{\pi}\left\{\left[(2x+3)\left(-\frac{1}{m}\right)\cos(mx)\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{2}{m}\,\cos(mx)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{4\pi}{m}\,\cos(m\pi) + \frac{2}{m^2}\,\big[\sin(mx)\big]_{-\pi}^{\pi}\right\} = (-1)^{m+1}\,\frac{4}{m}$$

$$\Rightarrow \quad u(x) \sim 3 + \sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{4}{n}\,\sin(nx).$$

7. For $m,\, n = 1, 2, \ldots,\; m \neq n$,

$$\int_a^b f_m f_n\,dx = \int_0^{\pi} \sin\frac{(2m-1)x}{2}\,\sin\frac{(2n-1)x}{2}\,dx$$

$$= \frac{1}{2}\int_0^{\pi}\big[\cos\big((m-n)x\big) - \cos\big((m+n-1)x\big)\big]\,dx$$

$$= \frac{1}{2}\left[\frac{1}{m-n}\,\sin\big((m-n)x\big) - \frac{1}{m+n-1}\,\sin\big((m+n-1)x\big)\right]_0^{\pi} = 0.$$

(i) We have

$$u(x) \sim \sum_{n=1}^{\infty} c_n \sin\frac{(2n-1)x}{2}$$

$$\Rightarrow \quad \int_0^{\pi} \sin\frac{(2m-1)x}{2}\,dx = c_m \int_0^{\pi} \sin^2\frac{(2m-1)x}{2}\,dx$$

$$= c_m \int_0^{\pi} \tfrac{1}{2}\big[1-\cos\big((2m-1)x\big)\big]\,dx = \tfrac{1}{2}\,\pi c_m$$

$$\Rightarrow \quad c_m = \frac{2}{\pi}\int_0^{\pi} \sin\frac{(2m-1)x}{2}\,dx$$

$$= -\frac{4}{(2m-1)\pi}\left[\cos\frac{(2m-1)x}{2}\right]_0^{\pi} = \frac{4}{(2m-1)\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad u(x) \sim \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)x}{2}.$$

(ii) As in (i), for $m = 1, 2, \ldots$

$$c_m = \frac{2}{\pi}\int_0^{\pi} (2x-1)\sin\frac{(2m-1)x}{2}\,dx$$

$$= \frac{2}{\pi}\Bigg\{\left[(2x-1)\left(-\frac{2}{2m-1}\right)\cos\frac{(2m-1)x}{2}\right]_0^{\pi} + \int_0^{\pi} \frac{4}{2m-1}\cos\frac{(2m-1)x}{2}\,dx\Bigg\}$$

$$= \frac{2}{\pi}\left\{-\frac{2}{2m-1} + \frac{8}{(2m-1)^2}\left[\sin\frac{(2m-1)x}{2}\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left[-\frac{2}{2m-1} + \frac{8}{(2m-1)^2}\sin\frac{(2m-1)\pi}{2}\right]$$

$$= -\frac{4}{(2m-1)\pi} + (-1)^{m+1}\frac{16}{(2m-1)^2\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad u(x) \sim \sum_{n=1}^{\infty} \left[-\frac{4}{(2n-1)\pi} + (-1)^{n+1} \frac{16}{(2n-1)^2\pi} \right] \sin \frac{(2n-1)x}{2}.$$

8. As above, for $m, n = 1, 2, \ldots, m \neq n$,

$$\int_a^b f_m f_n \, dx = \int_0^1 \cos \frac{(2m-1)\pi x}{2} \, \sin \frac{(2n-1)\pi x}{2} \, dx$$

$$= \frac{1}{2} \int_0^1 \left[\cos\big((m+n-1)\pi x\big) + \cos\big((m-n)\pi x\big) \right] dx$$

$$= \frac{1}{2} \left[\frac{1}{m+n-1} \, \sin\big((m+n-1)\pi x\big) + \frac{1}{m-n} \, \sin\big((m-n)\pi x\big) \right]_0^1 = 0.$$

(i) Here

$$u(x) \sim \sum_{n=1}^{\infty} c_n \cos \frac{(2n-1)\pi x}{2}$$

$$\Rightarrow \quad \int_0^1 \cos \frac{(2m-1)\pi x}{2} \, dx = c_m \int_0^1 \cos^2 \frac{(2m-1)\pi x}{2} \, dx$$

$$= c_m \int_0^1 \tfrac{1}{2} \left[1 + \cos\big((2m-1)\pi x\big) \right] dx = \tfrac{1}{2} \, c_m$$

$$\Rightarrow \quad c_m = 2 \int_0^{\pi} \cos \frac{(2m-1)\pi x}{2} \, dx$$

$$= \frac{4}{(2m-1)\pi} \left[\sin \frac{(2m-1)\pi x}{2} \right]_0^1 = (-1)^{m+1} \frac{4}{(2m-1)\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad u(x) \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{(2n-1)\pi} \cos \frac{(2n-1)\pi x}{2}.$$

(ii) As in (i),

$$
\begin{aligned}
c_m &= 2\int_0^1 (2x-1)\cos\frac{(2m-1)\pi x}{2}\,dx \\
&= 2\Bigg\{\bigg[(2x-1)\frac{2}{(2m-1)\pi}\,\sin\frac{(2m-1)\pi x}{2}\bigg]_0^1 \\
&\qquad\qquad - \int_0^1 \frac{4}{(2m-1)\pi}\,\sin\frac{(2m-1)\pi x}{2}\,dx\Bigg\} \\
&= 2\Bigg\{\frac{2}{(2m-1)\pi}\,\sin\frac{(2m-1)\pi}{2} + \frac{8}{(2m-1)^2\pi^2}\bigg[\cos\frac{(2m-1)\pi x}{2}\bigg]_0^1\Bigg\} \\
&= (-1)^{m+1}\frac{4}{(2m-1)\pi} - \frac{16}{(2m-1)^2\pi^2}, \quad m = 1, 2, \ldots
\end{aligned}
$$

$$
\Rightarrow \quad u(x) \sim \sum_{n=1}^{\infty}\bigg[(-1)^{n+1}\frac{4}{(2n-1)\pi} - \frac{16}{(2n-1)^2\pi^2}\bigg]\cos\frac{(2n-1)\pi x}{2}.
$$