Chapter 3

1. (i) This is a regular S-L problem with

$$p(x) \equiv 1, \quad q(x) \equiv 0, \quad \sigma(x) \equiv 1,$$
$$a = 0, \quad b = 1, \quad \kappa_1 = 1, \quad \kappa_2 = 2, \quad \kappa_3 = 0, \quad \kappa_4 = 1.$$

(ii) Since

$$f'' - f' + \lambda f = 0 \quad \Leftrightarrow \quad e^{-x} f'' - e^{-x} f' + \lambda e^{-x} f = 0$$
$$\Leftrightarrow \quad (e^{-x} f')' + \lambda e^{-x} f = 0,$$

this is a regular S-L problem with

$$p(x) = e^{-x}, \quad q(x) \equiv 0, \quad \sigma(x) = e^{-x},$$
$$a = 0, \quad b = 2, \quad \kappa_1 = 0, \quad \kappa_2 = 1, \quad \kappa_3 = 1, \quad \kappa_4 = -1.$$

2. (i) For m, $n = 1, 2, \ldots$, $m \neq n$,

$$\begin{aligned}
\int_a^b f_m f_n \, dx &= \int_0^\pi \sin(mx) \sin(nx) \, dx \\
&= \frac{1}{2} \int_0^\pi \big[\cos\big((m-n)x\big) - \cos\big((m+n)x\big) \big] \, dx \\
&= \frac{1}{2} \left[\frac{1}{m-n} \sin\big((m-n)x\big) - \frac{1}{m+n} \sin\big((m+n)x\big) \right]_0^\pi = 0.
\end{aligned}$$

(ii) For m, $n = 1, 2, \ldots$, $m \neq n$,

$$\int_a^b f_0 f_n \, dx = \int_0^\pi \cos(nx) \, dx = \frac{1}{n} \big[\sin(nx) \big]_0^\pi = 0,$$

$$\int_a^b f_m f_n \, dx = \int_0^\pi \cos(mx) \cos(nx) \, dx$$

$$= \frac{1}{2}\int_0^\pi \big[\cos\big((m+n)x\big) + \cos\big((m-n)x\big)\big]\,dx$$

$$= \frac{1}{2}\left[\frac{1}{m+n}\,\sin\big((m+n)x\big) + \frac{1}{m-n}\,\sin\big((m-n)x\big)\right]_0^\pi = 0.$$

(iii) As in (i) and (ii), for m, $n = 1, 2, \ldots$, $m \neq n$,

$$\int_a^b f_0 f_{1n}\,dx = \int_{-\pi}^{\pi} \cos(nx)\,dx = \frac{1}{n}\,\big[\sin(nx)\big]_{-\pi}^{\pi} = 0,$$

$$\int_a^b f_0 f_{2n}\,dx = \int_{-\pi}^{\pi} \sin(nx)\,dx = -\frac{1}{n}\,\big[\cos(nx)\big]_{-\pi}^{\pi} = 0,$$

$$\int_a^b f_{1m} f_{1n}\,dx = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx$$

$$= \frac{1}{2}\left[\frac{1}{m+n}\,\sin\big((m+n)x\big) + \frac{1}{m-n}\,\sin\big((m-n)x\big)\right]_{-\pi}^{\pi} = 0,$$

$$\int_a^b f_{2m} f_{2n}\,dx = \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx$$

$$= \frac{1}{2}\left[\frac{1}{m-n}\,\sin\big((m-n)x\big) - \frac{1}{m+n}\,\sin\big((m+n)x\big)\right]_{-\pi}^{\pi} = 0,$$

$$\int_a^b f_{1m} f_{2n}\,dx = \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \big[\sin\big((n+m)x\big) + \sin\big((n-m)x\big)\big]\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{n+m}\,\cos\big((n+m)x\big) - \frac{1}{n-m}\,\cos\big((n-m)x\big)\right]_{-\pi}^{\pi} = 0;$$

also, for $m = n$

$$\int_a^b f_{1n} f_{2n}\, dx = \int_{-\pi}^{\pi} \cos(nx)\sin(nx)\, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin(2nx)\, dx = -\frac{1}{4n}\left[\cos(2nx)\right]_{-\pi}^{\pi} = 0.$$

3. (i) We have

$$p(x) \equiv 1, \quad q(x) \equiv 0, \quad \sigma(x) \equiv 1, \quad a = 0, \quad b = \pi$$

$$\Rightarrow \quad \int_0^{\pi} \left[p(f')^2 - qf^2\right] dx = \int_0^{\pi} (f')^2\, dx \geq 0,$$

$$\left[pff'\right]_0^{\pi} = 0, \quad \int_0^{\pi} \sigma f^2\, dx = \int_0^{\pi} f^2\, dx > 0$$

$$\Rightarrow \quad \lambda = \frac{\int_0^{\pi} (f')^2\, dx}{\int_0^{\pi} f^2\, dx} \geq 0,$$

$$\lambda = 0 \quad \Leftrightarrow \quad f'(x) \equiv 0 \quad \Leftrightarrow \quad f(x) \equiv \text{const} = 0: \ \text{unacceptable}$$

$$\Rightarrow \quad \lambda > 0$$

$$\Rightarrow \quad f(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

$$\Rightarrow \quad f'(x) = -C_1\sqrt{\lambda}\, \sin(\sqrt{\lambda}\, x) + C_2\sqrt{\lambda}\, \cos(\sqrt{\lambda}\, x),$$

$$f(0) = 0 \quad \Rightarrow \quad C_1 = 0,$$

$$f'(\pi) = 0 \quad \Rightarrow \quad C_2\sqrt{\lambda}\, \cos(\sqrt{\lambda}\, \pi) = 0 \quad \Rightarrow \quad \cos(\sqrt{\lambda}\, \pi) = 0$$

$$\Rightarrow \quad \sqrt{\lambda}\, \pi = \frac{(2n-1)\pi}{2} \quad \Rightarrow \quad \lambda_n = \frac{(2n-1)^2}{4}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad f_n(x) = \sin\frac{(2n-1)x}{2}, \quad n = 1, 2, \ldots .$$

(ii) The proof that $\lambda > 0$ is almost identical to that in (i), with $b = 1$. Hence, with f and

f' also as in (i), we have

$$
\begin{aligned}
& f'(0) = 0 \quad \Rightarrow \quad C_2\sqrt{\lambda} = 0 \quad \Rightarrow \quad C_2 = 0, \\
& f(1) = 0 \quad \Rightarrow \quad C_1 \cos\sqrt{\lambda} = 0 \quad \Rightarrow \quad \cos\sqrt{\lambda} = 0 \\
\Rightarrow \quad & \sqrt{\lambda} = \frac{(2n-1)\pi}{2} \quad \Rightarrow \quad \lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad n = 1, 2, \dots \\
\Rightarrow \quad & f_n(x) = \cos\frac{(2n-1)\pi x}{2}, \quad n = 1, 2, \dots .
\end{aligned}
$$

(iii) Noting that here

$$
\left[pff'\right]_0^1 = f(1)f'(1) - f(0)f'(0) = -f^2(0) \leq 0,
$$

we repeat the argument in Example 3.18 and conclude that $\lambda > 0$. Then

$$
\begin{aligned}
& f(x) = C_1 \cos(\sqrt{\lambda}\,x) + C_2 \sin(\sqrt{\lambda}\,x) \\
\Rightarrow \quad & f'(x) = -C_1\sqrt{\lambda}\,\sin(\sqrt{\lambda}\,x) + C_2\sqrt{\lambda}\,\cos(\sqrt{\lambda}\,x), \\
& f(0) - f'(0) = 0 \quad \Rightarrow \quad C_1 = \sqrt{\lambda}\,C_2, \\
& f(1) = 0 \quad \Rightarrow \quad C_1 \cos\sqrt{\lambda} + C_2 \sin\sqrt{\lambda} = 0 \\
\Rightarrow \quad & \sqrt{\lambda}\,C_2 \cos\sqrt{\lambda} + C_2 \sin\sqrt{\lambda} = 0 \quad \Rightarrow \quad C_2(\sqrt{\lambda}\,\cos\sqrt{\lambda} + \sin\sqrt{\lambda}) = 0 \\
\Rightarrow \quad & \sqrt{\lambda}\,\cos\sqrt{\lambda} + \sin\sqrt{\lambda} = 0.
\end{aligned}
$$

Clearly, $\cos\sqrt{\lambda} \neq 0$; hence, $\tan\sqrt{\lambda} = -\sqrt{\lambda}$. This means that if ζ_n are the positive roots of the transcendental equation $\tan\zeta = -\zeta$, then (given that $C_1 = \sqrt{\lambda}\,C_2$) the eigenvalue-eigenfunction pairs are

$$
\lambda_n = \zeta_n^2, \quad f_n(x) = \zeta_n \cos(\zeta_n x) + \sin(\zeta_n x), \quad n = 1, 2, \dots .
$$

(iv) This time we have

$$
\left[pff'\right]_0^1 = f(1)f'(1) - f(0)f'(0) = -f^2(1) \leq 0,
$$

so the argument in Example 3.18 can be applied again to yield $\lambda > 0$. Therefore, with f and f' the same as in (iii), we have

$$
\begin{aligned}
& f'(0) = 0 \quad \Rightarrow \quad \sqrt{\lambda}\,C_2 = 0 \quad \Rightarrow \quad C_2 = 0, \\
& f(1) + f'(1) = 0 \quad \Rightarrow \quad C_1 \cos\sqrt{\lambda} - C_1\sqrt{\lambda}\,\sin\sqrt{\lambda} = 0.
\end{aligned}
$$

Since, as is easily seen, $C_1 \neq 0$ and $\sin\sqrt{\lambda} \neq 0$, we have $\cot\sqrt{\lambda} = \sqrt{\lambda}$. If ζ_n are the positive roots of the equation $\cot\zeta = \zeta$, then the eigenvalue-eigenfunction pairs are

$$\lambda_n = \zeta_n^2, \quad f_n(x) = \cos(\zeta_n x), \quad n = 1, 2, \ldots.$$

(v) Since

$$\big[pff'\big]_0^1 = f(1)f'(1) - f(0)f'(0) = -f^2(0) \leq 0,$$

we deduce as above that $\lambda > 0$. Hence, with f and f' once again as in (iii),

$$\begin{aligned}
&f(0) - f'(0) = 0 \quad \Rightarrow \quad C_1 = \sqrt{\lambda}\, C_2,\\
&f'(1) = 0 \quad \Rightarrow \quad -C_1\sqrt{\lambda}\, \sin\sqrt{\lambda} + C_2\sqrt{\lambda}\, \cos\sqrt{\lambda} = 0\\
\Rightarrow \quad & -C_1\sqrt{\lambda}\, \sin\sqrt{\lambda} + C_1 \cos\sqrt{\lambda} = 0 \quad \Rightarrow \quad C_1(-\sqrt{\lambda}\, \sin\sqrt{\lambda} + \cos\sqrt{\lambda}) = 0.
\end{aligned}$$

As above, we cannot have $C_1 = 0$ nor $\sin\sqrt{\lambda} = 0$; therefore, $\cot\sqrt{\lambda} = \sqrt{\lambda}$. Let ζ_n be the positive roots of the equation $\cot\zeta = \zeta$. Then, since $C_1 = \sqrt{\lambda}\, C_2$, the eigenvalue-eigenfunction pairs are

$$\lambda_n = \zeta_n^2, \quad f_n(x) = \zeta_n \cos(\zeta_n x) + \sin(\zeta_n x), \quad n = 1, 2, \ldots.$$

(vi) From the characteristic equation:

$$s^2 - s + \lambda = 0 \quad \Rightarrow \quad s_1 = \tfrac{1}{2}(1 + \sqrt{1 - 4\lambda}), \quad s_2 = \tfrac{1}{2}(1 - \sqrt{1 - 4\lambda}).$$

If $\lambda < 1/4$, then s_1 and s_2 are real and distinct, so

$$f(x) = C_1 e^{s_1 x} + C_2 e^{s_2 x}, \quad f'(x) = c_1 s_1 e^{s_1 x} + C_2 s_2 e^{s_2 x}.$$

Then the BCs yield the homogeneous linear algebraic system

$$\begin{aligned}
C_1 s_1 + C_2 s_2 &= 0,\\
C_1 s_1 e^{s_1} + C_2 s_2 e^{s_2} &= 0
\end{aligned}$$

with determinant

$$\begin{vmatrix} s_1 & s_2 \\ s_1 e^{s_1} & s_2 e^{s_2} \end{vmatrix} = s_1 s_2 (e^{s_1} - e^{s_2}).$$

Clearly, $s_1 \neq 0$ and $e^{s_1} \neq e^{s_2}$, but $s_2 = 0$ if $\lambda = 0$. In the latter case, $f(x) = C_1 e^x + C_2$, and the BCs imply that $C_1 = 0$. Consequently, we have the eigenvalue-eigenfunction pair

$$\lambda_0 = 0, \quad f_0(x) \equiv 1.$$

If $\lambda = 1/4$, then $s_1 = s_2 = 1/2$, so

$$f(x) = (C_1 + C_2x)e^{x/2}, \quad f'(x) = \big(C_2 + \tfrac{1}{2}\,C_1 + \tfrac{1}{2}\,C_2x\big)e^{x/2},$$

and the BCs yield the system

$$\begin{aligned} C_1 + \tfrac{1}{2}\,C_2 &= 0,\\ \tfrac{1}{2}\,C_1 + \tfrac{3}{2}\,C_2 &= 0, \end{aligned}$$

which has the unique (unacceptable) solution $C_1 = C_2 = 0$.

If $\lambda > 1/4$, then

$$\begin{aligned} & f(x) = e^{x/2}\big[C_1\cos\big(\tfrac{1}{2}\sqrt{4\lambda-1}\,x\big) + C_2\sin\big(\tfrac{1}{2}\sqrt{4\lambda-1}\,x\big)\big]\\ \Rightarrow\quad & f'(x) = e^{x/2}\big[\tfrac{1}{2}(C_1 + C_2\sqrt{4\lambda-1})\cos\big(\tfrac{1}{2}\sqrt{4\lambda-1}\,x\big)\\ &\qquad\qquad + \tfrac{1}{2}(C_2 - C_1\sqrt{4\lambda-1})\sin\big(\tfrac{1}{2}\sqrt{4\lambda-1}\,x\big)\big], \end{aligned}$$

and the BCs lead to

$$C_1 + C_2\sqrt{4\lambda-1} = 0, \quad (C_2 - C_1\sqrt{4\lambda-1})\sin\big(\tfrac{1}{2}\sqrt{4\lambda-1}\big) = 0.$$

If $C_2 - C_1\sqrt{4\lambda-1} = 0$, then C_1 and C_2 would satisfy a homogeneous linear system with determinant $4\lambda \neq 0$, which would imply that $C_1 = C_2 = 0$; therefore,

$$\begin{aligned} & \sin\big(\tfrac{1}{2}\sqrt{4\lambda-1}\big) = 0 \quad \Rightarrow \quad \tfrac{1}{2}\sqrt{4\lambda-1} = n\pi\\ \Rightarrow\quad & \lambda_n = \tfrac{1}{4}(4n^2\pi^2 + 1) = n^2\pi^2 + \tfrac{1}{4}, \quad n = 1, 2, \dots. \end{aligned}$$

The equality $C_1 + C_2\sqrt{4\lambda-1} = 0$ now reduces to $C_1 + 2C_2n\pi = 0$, so, taking $C_1 = -2n\pi$ and $C_2 = 1$, we obtain the eigenfunctions

$$f_n(x) = e^{x/2}\big[-2n\pi\cos(n\pi x) + \sin(n\pi x)\big], \quad n = 1, 2, \dots.$$

4. (i) For $n = 1, 2, \dots,$

$$\int_a^b u f_n\,dx = \int_0^\pi \sin(nx)\,dx = -\frac{1}{n}\big[\cos(nx)\big]_0^\pi = \frac{1}{n}\big[1 - (-1)^n\big],$$

$$\int_a^b f_n^2\,dx = \int_0^\pi \sin^2(nx)\,dx = \tfrac{1}{2}$$

$$\Rightarrow \quad c_n = \frac{1}{n}\big[1-(-1)^n\big]\cdot\frac{2}{\pi} = \frac{2}{n\pi}\,\big[1-(-1)^n\big]$$

$$\Rightarrow \quad u(x) \sim \sum_{n=1}^{\infty}\big[1-(-1)^n\big]\,\frac{2}{n\pi}\,\sin(nx).$$

(ii) By direct calculation, for $n = 1, 2, \ldots$ we have

$$\int_a^b u f_n\,dx = \int_0^\pi (2x+3)\sin(nx)\,dx$$

$$= \left[(2x+3)\left(-\frac{1}{n}\right)\cos(nx)\right]_0^\pi + \int_0^\pi \frac{2}{n}\,\cos(nx)\,dx$$

$$= \frac{3}{n} - \frac{2\pi+3}{n}\,\cos(n\pi) + \frac{2}{n^2}\,\big[\sin(nx)\big]_0^\pi = \big[3-(2\pi+3)(-1)^n\big]\,\frac{1}{n}$$

$$\Rightarrow \quad c_n = \frac{1}{n}\,\big[3-(2\pi+3)(-1)^n\big]\,\frac{2}{\pi}$$

$$= \frac{2}{n\pi}\,\big[3-(2\pi+3)(-1)^n\big]$$

$$\Rightarrow \quad u(x) \sim \sum_{n=1}^{\infty}\frac{2}{n\pi}\,\big[3-(2\pi+3)(-1)^n\big]\sin(nx).$$

(iii) With $L = 1$, for $n = 1, 2, \ldots$ we have

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$= 2\left[\int_0^{1/2} x\sin(n\pi x)\,dx + \int_{1/2}^{1}(1-x)\sin(n\pi x)\,dx\right]$$

$$= 2\Big\{\Big[x\Big(-\frac{1}{n\pi}\Big)\cos(n\pi x)\Big]_0^{1/2} + \int_0^{1/2} \frac{1}{n\pi}\cos(n\pi x)\,dx$$

$$+ \Big[(1-x)\Big(-\frac{1}{n\pi}\Big)\cos(n\pi x)\Big]_{1/2}^{1} + \int_{1/2}^{1} \frac{1}{n\pi}\cos(n\pi x)\,(-dx)\Big\}$$

$$= 2\Big\{-\frac{1}{2n\pi}\cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2}\big[\sin(n\pi x)\big]_0^{1/2}$$

$$+ \frac{1}{2n\pi}\cos\frac{n\pi}{2} - \frac{1}{n^2\pi^2}\big[\sin(n\pi x)\big]_{1/2}^{1}\Big\}$$

$$= \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow \quad u(x) \sim \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\sin(n\pi x).$$

5. (i) Since $f(x) \equiv 1$ is an eigenfunction, the expansion for $u(x) \equiv 1$ coincides with the function itself. This can also be verified by direct calculation:

$$\int_a^b u f_0\,dx = \int_0^\pi dx = \pi,$$

$$\int_a^b u f_n\,dx = \int_0^\pi \cos(nx)\,dx = \frac{1}{n}\big[\sin(nx)\big]_0^\pi = 0,$$

$$\int_0^\pi f_0^2\,dx = \int_0^\pi dx = \pi$$

$$\Rightarrow \quad c_0 = \pi \cdot \frac{1}{\pi} = 1, \quad c_n = 0, \quad n = 1, 2, \ldots.$$

(ii) Here

$$\int_a^b u f_0\,dx = \int_0^\pi (2x+3)\,dx = \big[x^2 + 3x\big]_0^\pi = \pi^2 + 3\pi,$$

$$\int_a^b u f_n \, dx = \int_0^\pi (2x+3)\cos(nx)\, dx$$

$$= \left[(2x+3)\frac{1}{n}\ \sin(nx)\right]_0^\pi - \int_0^\pi \frac{2}{n}\ \sin(nx)\, dx$$

$$= \frac{2}{n^2}\left[\cos(nx)\right]_0^\pi = \left[(-1)^n - 1\right]\frac{2}{n^2}, \quad n = 1, 2, \ldots,$$

$$\int_0^\pi f_0^2\, dx = \pi, \quad \int_0^\pi f_n^2\, dx = \frac{\pi}{2}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad c_0 = (\pi^2 + 3\pi)\,\frac{1}{\pi} = \pi + 3,$$

$$c_n = \left[(-1)^n - 1\right]\frac{2}{n^2}\,\frac{2}{\pi} = \left[(-1)^n - 1\right]\frac{4}{n^2\pi}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x) \sim \pi + 3 + \sum_{n=1}^{\infty}\left[(-1)^n - 1\right]\frac{4}{n^2\pi}\ \cos(nx).$$

(iii) For $L = 1$,

$$a_0 = \int_0^1 f(x)\, dx = \int_0^{1/2} x\, dx + \int_{1/2}^1 (1-x)\, dx = \tfrac{1}{2}\left[x^2\right]_0^{1/2} + \left[x - \tfrac{1}{2}\,x^2\right]_{1/2}^1 = \tfrac{1}{4},$$

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\, dx$$

$$= 2\left[\int_0^{1/2} x\cos(n\pi x)\, dx + \int_{1/2}^1 (1-x)\cos(n\pi x)\, dx\right]$$

$$= 2\left\{\left[x\,\frac{1}{n\pi}\ \sin(n\pi x)\right]_0^{1/2} - \int_0^{1/2}\frac{1}{n\pi}\sin(n\pi x)\, dx\right.$$

$$+\left[(1-x)\frac{1}{n\pi}\,\sin(n\pi x)\right]_{1/2}^{1}-\int\limits_{1/2}^{1}\frac{1}{n\pi}\,\sin(n\pi x)\,(-dx)\Bigg\}$$

$$=2\Bigg\{\frac{1}{2n\pi}\,\sin\frac{n\pi}{2}+\frac{1}{n^2\pi^2}\,\big[\cos(n\pi x)\big]_0^{1/2}$$

$$-\frac{1}{2n\pi}\,\sin\frac{n\pi}{2}-\frac{1}{n^2\pi^2}\,\big[\cos(n\pi x)\big]_{1/2}^{1}\Bigg\}$$

$$=2\left[\frac{1}{n^2\pi^2}\,\cos\frac{n\pi}{2}-\frac{1}{n^2\pi^2}-\frac{1}{n^2\pi^2}\,\cos(n\pi)+\frac{1}{n^2\pi^2}\,\cos\frac{n\pi}{2}\right]$$

$$=2\left[\frac{2}{n^2\pi^2}\,\cos\frac{n\pi}{2}+(-1)^{n+1}\,\frac{1}{n^2\pi^2}-\frac{1}{n^2\pi^2}\right],\quad n=1,2,\dots$$

$$\Rightarrow\quad u(x)\sim\frac{1}{4}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left[2\cos\frac{n\pi}{2}+(-1)^{n+1}-1\right]\cos(n\pi x).$$