5. The standard asymptotic expansion for u leads to

$$u_0 = e^{-t},\ \ u_1 = -(u_0)_t + (u_0)_{xx} = e^{-t}, \ldots,\ \ u_n = e^{-t},\ n = 0, 1, 2, \ldots \ \text{(by induction)};$$

hence,

$$u(x,t) = e^{-t}(1 + \varepsilon + \varepsilon^2 + \cdots) = \frac{e^{-t}}{1-\varepsilon},$$

which does not satisfy the IC. Consequently, we introduce a boundary layer near $t = 0$ that, by the argument used in Example 13.13, should be of width ε, yielding the substitution $\tau = t/\varepsilon$, $u(x,t) = u(x,\varepsilon\tau) = v(x,\tau)$. The IVP now becomes

$$v_\tau - \varepsilon v_{xx} + v = e^{-\varepsilon\tau} = 1 - \varepsilon\tau + O(\varepsilon^2), \quad v(x,0) = \sin x,$$

and the asymptotic expansion procedure applied to v leads to

$$\begin{aligned}
&(v_0)_\tau + v_0 = 1, \quad v_0(x,0) = \sin x \\
\Rightarrow\ & v_0 = Ce^{-\tau} + 1 = (\sin x - 1)e^{-\tau} + 1, \\
&(v_1)_\tau + v_1 = (v_0)_{xx} - \tau = -e^{-\tau}\sin x - \tau, \quad v_1(x,0) = 0 \\
\Rightarrow\ & v_1 = Ce^{-\tau} - \tau e^{-\tau}\sin x + 1 - \tau = -e^{-\tau} - \tau e^{-\tau}\sin x + 1 - \tau \\
\Rightarrow\ & v(x,t) = (\sin x - 1)e^{-\tau} + 1 + \varepsilon(-e^{-\tau} - \tau e^{-\tau}\sin x + 1 - \tau) + O(\varepsilon^2).
\end{aligned}$$

Applying the matching procedure described in Example 13.12, we have

$$\begin{aligned}
u^o(x,t) &= u(x,t) = \frac{e^{-t}}{1-\varepsilon} = \frac{e^{-\varepsilon\tau}}{1-\varepsilon} = (1 - \varepsilon\tau + O(\varepsilon^2))(1 + \varepsilon + O(\varepsilon^2)) \\
&= 1 + \varepsilon - \varepsilon\tau + O(\varepsilon^2) = 1 - t + \varepsilon + O(\varepsilon^2), \\
u^i(x,t) &= v(x,t/\varepsilon) = (\sin x - 1)e^{-t/\varepsilon} + 1 \\
&\qquad + \varepsilon\Big(- e^{-t/\varepsilon} - \frac{t}{\varepsilon}\, e^{-t/\varepsilon}\sin x + 1 - \frac{t}{\varepsilon}\Big) + O(\varepsilon^2) \\
&= 1 - t + \varepsilon + O(\varepsilon^2) \\
\Rightarrow\ (u^o)^i &= (u^i)^o = 1 - t + \varepsilon + O(\varepsilon^2),
\end{aligned}$$

so the inner and outer solutions are matched up to $O(\varepsilon)$ terms and the composite solution is

$$\begin{aligned}
u^c(x,t) &= 1 - t + \varepsilon + (\sin x - 1)e^{-t/\varepsilon} - te^{-t/\varepsilon}\sin x - \varepsilon e^{-t/\varepsilon} + O(\varepsilon^2) \\
&= 1 - t + \varepsilon + \big[(1-t)\sin x - 1 - \varepsilon\big]e^{-t/\varepsilon} + O(\varepsilon^2).
\end{aligned}$$

6. An expansion of the form $u(x,t) = u_0(x,t) + O(\varepsilon)$ yields

$$(u_0)_y - 2(u_0)_x = 0, \quad u_0(x,0) = 1, \quad u_0(x,1) = \sin x.$$

Using the method of characteristics with $y = 1$ as data line, we have

$$\begin{aligned} & x'(y) = -2, \quad x(1) = x_0 \quad \Rightarrow \quad x = -2y + x_0 + 2 \quad \Rightarrow \quad x_0 = x + 2y - 2 \\ \Rightarrow \quad & \frac{du_0}{dy} = 0 \quad \Rightarrow \quad u_0(x,y) = C = u_0(x_0,1) = \sin x_0 = \sin(x+2y-2) \\ \Rightarrow \quad & u(x,y) = \sin(x+2y-2) + O(\varepsilon), \end{aligned}$$

which does not satisfy the BC at $y = 0$. As in Example 13.13, we determine that the boundary layer near $y = 0$ should be of width ε, so we set

$$\eta = y/\varepsilon, \quad u(x,y) = u(x,\varepsilon\eta) = v(x,\eta)$$

and arrive at the new problem

$$\varepsilon^2 v_{xx} + v_{\eta\eta} - 2\varepsilon v_x + v_\eta = 0, \quad v(x,0) = 0.$$

Here the expansion $v(x,\eta) = v_0(x,\eta) + O(\varepsilon)$ leads to

$$\begin{aligned} & (v_0)_{\eta\eta} + (v_0)_\eta = 0, \quad v_0(x,0) = 0 \\ \Rightarrow \quad & v_0 = C_1(x) + C_2(x)e^{-\eta} = 1 - C_2(x) + C_2(x)e^{-\eta} \\ \Rightarrow \quad & v(x,\eta) = 1 - C_2(x) + C_2(x)e^{-\eta} + O(\varepsilon); \end{aligned}$$

hence, matching the inner and outer solutions, we obtain

$$\begin{aligned} & u^o(x,y) = u(x,y) = \sin(x+2y-2) + O(\varepsilon) \\ \Rightarrow \quad & (u^o)^i = \sin(x+2\varepsilon\eta-2) + O(\varepsilon) = \sin(x-2) + O(\varepsilon), \\ & u^i(x,y) = v(x,y/\varepsilon) = 1 - C_2(x) + C_2(x)e^{-y/\varepsilon} + O(\varepsilon) \\ \Rightarrow \quad & (u^i)^o = 1 - C_2(x) + O(\varepsilon) \\ \Rightarrow \quad & C_2(x) = 1 - \sin(x-2) \\ \Rightarrow \quad & u^i(x,y) = \sin(x-2) + \big[1 - \sin(x-2)\big]e^{-y/\varepsilon} + O(\varepsilon) \\ \Rightarrow \quad & u^c(x,y) = \sin(x+2y-2) + \big[1 - \sin(x-2)\big]e^{-y/\varepsilon} + O(\varepsilon). \end{aligned}$$

7. The asymptotic expansion $u(x,y) = u_0(x,y) + O(\varepsilon)$ yields the BVP

$$\begin{aligned}
&(u_0)_x - 2(u_0)_y = 0, \quad 0 < x < 1,\ y > 0,\\
&u_0(x,0) = 1 - x, \quad 0 < x < 1,\\
&u_0(0,y) = p(y) = \begin{cases} y-2, & 0 < y \le 2,\\ 0, & y > 2,\end{cases}\\
&u_0(1,y) = e^{-2y}, \quad y > 0.
\end{aligned}$$

First, we use the method of characteristics on the PDE with $x = 0$ as data line:

$$\begin{aligned}
&y'(x) = -2, \quad y(0) = y_0 \quad \Rightarrow \quad y = -2x + y_0 \quad \Rightarrow \quad y_0 = y + 2x\\
\Rightarrow \quad &\frac{du_0}{dx} = 0 \quad \Rightarrow \quad u_0(x,y) = C = u_0(0,y_0) = p(y_0) = p(y+2x)\\
\Rightarrow \quad &u(x,y) = p(y+2x) + O(\varepsilon).
\end{aligned}$$

This function does not satisfy the BCs on $y = 0$ and $x = 1$, so we need to introduce two boundary layers. As in Example 13.13, the analysis of the layer width reveals that in both cases this should be ε.

(i) Near $y = 0$ we set

$$\eta = y/\varepsilon, \quad u(x,y) = u(x,\varepsilon\eta) = v(x,\eta)$$

and, expanding v in an asymptotic series, obtain

$$\begin{aligned}
&\varepsilon^2 v_{xx} + v_{\eta\eta} + \varepsilon v_x - 2v_\eta = 0, \quad v(x,0) = 1 - x\\
\Rightarrow \quad &(v_0)_{\eta\eta} + 2(v_0)_\eta = 0, \quad v_0(x,0) = 1 - x\\
\Rightarrow \quad &v_0(x,\eta) = C_1(x) + C_2(x)e^{-2\eta} = (1 - x - C_2(x)) + C_2(x)e^{-2\eta}\\
\Rightarrow \quad &v(x,\eta) = (1 - x - C_2(x)) + C_2(x)e^{-2\eta} + O(\varepsilon).
\end{aligned}$$

(ii) Near $x = 1$ we set

$$\xi = (1-x)/\varepsilon, \quad u(x,y) = u(1 - \varepsilon\xi, y) = w(\xi, y)$$

and, as in (i), find that

$$\begin{aligned}
&w_{\xi\xi} + \varepsilon^2 w_{yy} + w_\xi + 2\varepsilon w_y = 0, \quad w(0,y) = e^{-2y}\\
\Rightarrow \quad &(w_0)_{\xi\xi} + (w_0)_\xi = 0, \quad w_0(0,y) = e^{-2y}\\
\Rightarrow \quad &w_0(\xi,y) = D_1(y) + D_2(y)e^{-\xi} = e^{-2y} - D_2(y) + D_2(y)e^{-\xi}\\
\Rightarrow \quad &w(\xi,y) = e^{-2y} - D_2(y) + D_2(y)e^{-\xi} + O(\varepsilon).
\end{aligned}$$

Then the matching procedure (see Example 13.12) for each of the two boundary layers yields

$$
\begin{aligned}
& u^o(x,y) = u(x,y) = p(y+2x) + O(\varepsilon) \\
\Rightarrow\quad & (u^o)^{i_1} = p(\varepsilon\eta + 2x) + O(\varepsilon) = p(2x) + O(\varepsilon), \\
& u^{i_1}(x,y) = v(x, y/\varepsilon) = (1 - x - C_2(x)) + C_2 e^{-2y/\varepsilon} + O(\varepsilon) \\
\Rightarrow\quad & (u^{i_1})^o(x,y) = 1 - x - C_2(x) + O(\varepsilon), \\
& (u^o)^{i_1}(x,y) = (u^{i_1})^o(x,y) \quad\Rightarrow\quad C_2(x) = 1 - x - p(2x) \\
\Rightarrow\quad & u^{i_1}(x,y) = p(2x) + \big[1 - x - p(2x)\big]e^{-2y/\varepsilon} + O(\varepsilon), \\
& (u^o)^{i_2} = p(y + 2 - 2\varepsilon\xi) + O(\varepsilon) = p(y+2) + O(\varepsilon), \\
& u^{i_2}(x,y) = w((x-1)/\varepsilon, y) = e^{-2y} - D_2(y) + D_2(y)e^{(x-1)/\varepsilon} + O(\varepsilon) \\
\Rightarrow\quad & (u^{i_2})^o = e^{-2y} - D_2(y) + O(\varepsilon), \\
& (u^o)^{i_2} = (u^{i_2})^o \quad\Rightarrow\quad D_2(y) = e^{-2y} - p(y+2) \\
\Rightarrow\quad & u^{i_2}(x,y) = p(y+2) + \big[e^{-2y} - p(y+2)\big]e^{(x-1)/\varepsilon} + O(\varepsilon).
\end{aligned}
$$

Since $p(2x) = 2x - 2$, $0 < x < 1$, and $p(y+2) = 0$, $y > 0$, we finally have

$$
\begin{aligned}
& u^{i_1}(x,y) = 2(x-1) + 3(1-x)e^{-2y/\varepsilon} + O(\varepsilon), \\
& u^{i_2}(x,y) = e^{-2y}e^{(x-1)/\varepsilon} + O(\varepsilon) = e^{-2y+(x-1)/\varepsilon} + O(\varepsilon) \\
\Rightarrow\quad & u^c(x,y) = u^o(x,y) + u^{i_1}(x,y) + u^{i_2}(x,y) - (u^o)^{i_1}(x,y) - (u^o)^{i_2}(x,y) \\
& \qquad = p(y+2x) + 3(1-x)e^{-2y/\varepsilon} + e^{-2y+(x-1)/\varepsilon} + O(\varepsilon) \\
& \qquad = \begin{cases} y + 2x - 2 + 3(1-x)e^{-2y/\varepsilon} + e^{-2y+(x-1)/\varepsilon} + O(\varepsilon), & y < 2(1-x), \\ 3(1-x)e^{-2y/\varepsilon} + e^{-2y+(x-1)/\varepsilon} + O(\varepsilon), & y \geq 2(1-x). \end{cases}
\end{aligned}
$$