

4. The IVPs satisfied by the coefficients  $u_0$  and  $u_1$  in the asymptotic expansion for  $u$  are determined in the usual way and are solved by the method of characteristics:

$$\begin{aligned}
& (u_0)_t + (u_0)_x = 0, \quad u_0(x, 0) = \cos x \\
\Rightarrow & x'(t) = 1, \quad x(0) = x_0 \quad \Rightarrow \quad x = t + x_0 \quad \Rightarrow \quad x_0 = x - t \\
\Rightarrow & \frac{du_0}{dt} = 0 \quad \Rightarrow \quad u_0(x, t) = C = u_0(x_0, 0) = \cos x_0 = \cos(x - t), \\
& (u_1)_t + (u_1)_x = -u_0 = -\cos(x - t) \\
\Rightarrow & x'(t) = 1, \quad x(0) = x_0, \quad \Rightarrow \quad x = t + x_0 \quad \Rightarrow \quad x_0 = x - t \\
\Rightarrow & \frac{du_1}{dt} = -\cos(x - t) = -\cos x_0 \\
\Rightarrow & u_1(x, t) = -t \cos x_0 = -t \cos(x - t) \\
\Rightarrow & u(x, t) = \cos(x - t) - \varepsilon t \cos(x - t) + O(\varepsilon^2).
\end{aligned}$$

If  $t = O(\varepsilon^{-1})$ , then  $\varepsilon t = O(1)$ , so we have secular terms. To make the series valid for all  $t$ , we set

$$\tau = \varepsilon t, \quad u(x, t) = v(x, t, \tau) \quad \Rightarrow \quad u(x, 0) = v(x, 0, 0).$$

Consequently,  $u_t = v_t + \varepsilon v_\tau$ , so the new IVP is

$$v_t + \varepsilon v_\tau + v_x + \varepsilon v = 0, \quad v(x, 0, 0) = \cos x.$$

The coefficient  $v_0$  in the asymptotic expansion for  $v$  satisfies the IVP

$$(v_0)_t + (v_0)_x = 0, \quad v_0(x, 0, 0) = \cos x.$$

Since this PDE is linear, homogeneous, and independent of  $\tau$ , its general solution is of the form  $v_0(x, t, \tau) = \varphi(\tau)f(x, t)$ , where  $f$  satisfies the same PDE as  $v_0$ ,  $f(x, 0) = \cos x$ , and  $\varphi(\tau)$  is arbitrary with  $\varphi(0) = 1$ . By comparison with the solution for  $u_0$ , we therefore have

$$v_0(x, t, \tau) = \varphi(\tau) \cos(x - t), \quad \varphi(0) = 1.$$

Then the IVP for  $v_1$  is

$$(v_1)_t + (v_1)_x = -(v_0)_\tau - v_0 = [-\varphi'(\tau) - \varphi(\tau)] \cos(x - t), \quad v_1(x, 0, 0) = 0.$$

The characteristics are the same as for  $u_1$ , that is,  $x = t + x_0$ , and on them (with  $\tau$  regarded as a parameter)

$$\frac{dv_1}{dt} = [-\varphi'(\tau) - \varphi(\tau)] \cos x_0;$$

hence,

$$v_1(x, t, \tau) = -t[\varphi'(\tau) + \varphi(\tau)] \cos x_0 = -t[\varphi'(\tau) + \varphi(\tau)] \cos(x - t).$$

Thus, we have

$$v(x, t, \tau) = \varphi(\tau) \cos(x - t) - \varepsilon t[\varphi'(\tau) + \varphi(\tau)] \cos(x - t) + O(\varepsilon^2).$$

To eliminate the secular term, we choose  $\varphi$  so that  $\varphi'(\tau) + \varphi(\tau) = 0$ . Since  $\varphi(0) = 1$ , we find that  $\varphi(\tau) = e^{-\tau}$ . Hence, the asymptotic expansion for the solution of the given IVP can now be written in the form

$$u(x, t) = v(x, t, \tau) = e^{-\tau} \cos(x - t) + O(\varepsilon^2) = e^{-\varepsilon t} \cos(x - t) + O(\varepsilon^2).$$

It is easily checked that  $e^{-\varepsilon t} \cos(x - t)$  is, in fact, the exact solution of the IVP.