

## Chapter 13

1. With the usual asymptotic series expansion for  $u$ , writing the Laplacian in terms of polar coordinates, noting that the solution depends only on the polar radius  $r$ , and using the integrating factor method, we find that

$$\begin{aligned}
 & ((u_0)_r)_r + r^{-1}(u_0)_r = 1, \quad u_0(0) \text{ bounded}, \quad u_0(1) = 1 \\
 \Rightarrow & (u_0)_r = r^{-1} \int r \, dr = r^{-1} \left( \frac{1}{2} r^2 + C_1 \right) = \frac{1}{2} r + C_1 r^{-1} \\
 \Rightarrow & u_0(r) = \frac{1}{4} r^2 + C_1 \ln r + C_2, \\
 & u_0(0) \text{ bounded} \Rightarrow C_1 = 0, \\
 & u_0(1) = 1 \Rightarrow 1 = \frac{1}{4} + C_2 \Rightarrow C_2 = \frac{3}{4} \Rightarrow u_0(r) = \frac{1}{4} r^2 + \frac{3}{4}, \\
 & ((u_1)_r)_r + r^{-1}(u_1)_r = -4u_0 = -r^2 - 3, \quad u_1(0) \text{ bounded}, \quad u_1(1) = 1 \\
 \Rightarrow & (u_1)_r = r^{-1} \int r(-r^2 - 3) \, dr = r^{-1} \left( -\frac{1}{4} r^4 - \frac{3}{2} r^2 + C_1 \right) = -\frac{1}{4} r^3 - \frac{3}{2} r + C_1 r^{-1} \\
 \Rightarrow & u_1(r) = -\frac{1}{16} r^4 - \frac{3}{4} r^2 + C_1 \ln r + C_2, \\
 & u_1(0) \text{ bounded} \Rightarrow C_1 = 0, \\
 & u_1(1) = 1 \Rightarrow 1 = -\frac{1}{16} - \frac{3}{4} + C_2 \Rightarrow C_2 = \frac{29}{16} \\
 \Rightarrow & u_1(r) = -\frac{1}{16} r^4 - \frac{3}{4} r^2 + \frac{29}{16} \\
 \Rightarrow & u(r) = \left( \frac{1}{4} r^2 + \frac{3}{4} \right) + \varepsilon \left( -\frac{1}{16} r^4 - \frac{3}{4} r^2 + \frac{29}{16} \right) + O(\varepsilon^2).
 \end{aligned}$$

2. As in (i), we have  $u = u(r)$ , so

$$\begin{aligned}
 & u^2 = (u_0 + \varepsilon u_1 + O(\varepsilon^2))^2 = u_0^2 + 2\varepsilon u_0 u_1 + O(\varepsilon^2) \\
 \Rightarrow & ((u_0)_r)_r + r^{-1}(u_0)_r = 36r, \quad u_0(0) \text{ bounded}, \quad u_0(1) = 4 \\
 \Rightarrow & (u_0)_r = r^{-1} \int r \cdot 36r \, dr = r^{-1} (12r^3 + C_1) = 12r^2 + C_1 r^{-1} \\
 \Rightarrow & u_0(r) = 4r^3 + C_1 \ln r + C_2, \\
 & u_0(0) \text{ bounded} \Rightarrow C_1 = 0, \\
 & u_0(1) = 4 \Rightarrow 4 = 4 + C_2 \Rightarrow C_2 = 0 \\
 \Rightarrow & u_0(r) = 4r^3,
 \end{aligned}$$

$$\begin{aligned}
& ((u_1)_r)_r + r^{-1}(u_1)_r = -u_0^2 = -16r^6, \quad u_1(0) \text{ bounded} \quad u_1(1) = 0 \\
\Rightarrow & (u_1)_r = r^{-1} \int r(-16r^6) dr = r^{-1}(-2r^8 + C_1) = -2r^7 + C_1 r^{-1} \\
\Rightarrow & u_1(r) = -\frac{1}{4}r^8 + C_1 \ln r + C_2, \\
& u_1(0) \text{ bounded} \quad \Rightarrow \quad C_1 = 0, \\
& u_1(1) = 0 \quad \Rightarrow \quad 0 = -\frac{1}{4} + C_2 \quad C_2 = \frac{1}{4} \\
\Rightarrow & u_1(r) = -\frac{1}{4}r^8 + \frac{1}{4} \\
\Rightarrow & u(r) = 4r^3 + \varepsilon \left( -\frac{1}{4}r^8 + \frac{1}{4} \right) + O(\varepsilon^2).
\end{aligned}$$

**3.** The general asymptotic expansion for  $u$  leads to

$$\begin{aligned}
& (u_0)_t + u_0 = 0, \quad u_0(x, 0) = \sin x \\
\Rightarrow & u_0(x, t) = C e^{-t} = e^{-t} \sin x, \\
& (u_1)_t + u_1 = (u_0)_x = e^{-t} \cos x, \quad u_1(x, 0) = 0 \\
\Rightarrow & u_1(x, t) = C e^{-t} + t e^{-t} \cos x = t e^{-t} \cos x, \\
& (u_2)_t + u_2 = (u_1)_x = -t e^{-t} \sin x, \quad u_2(x, 0) = 0 \\
\Rightarrow & u_2(x, t) = C e^{-t} - \frac{1}{2} t^2 e^{-t} \sin x = -\frac{1}{2} t^2 e^{-t} \sin x \\
\Rightarrow & u_n(x, t) = \frac{1}{n!} t^n e^{-t} (\sin x)^{(n)} \quad (\text{by induction}) \\
\Rightarrow & u(x, t) = e^{-t} \sum_{n=0}^{\infty} \frac{(\sin x)^{(n)}}{n!} (\varepsilon t)^n = e^{-t} \sin(x + \varepsilon t).
\end{aligned}$$

We note that, when truncated to any order of  $\varepsilon$ , the series is not properly arranged for  $t = O(\varepsilon^{-1})$  since all terms are  $O(1)$  (they are all secular terms). The final form of the solution above also reflects this feature.