5. (i) As in Section 12.3 (with $c = 1$),

$$
\begin{aligned}
& u(x,t) = F(x-t) + G(x+t) \quad \Rightarrow \quad u_t(x,t) = -F'(x-t) + G'(x+t) \\
\Rightarrow \quad & u(x,0) = F(x) + G(x) = x - 2 \quad \Rightarrow \quad F'(x) + G'(x) = 1, \\
& u_t(x,0) = -F'(x) + G'(x) = -2 \quad \Rightarrow \quad F'(x) = \tfrac{3}{2}, \quad G'(x) = -\tfrac{1}{2} \\
\Rightarrow \quad & F(x) = \tfrac{3}{2}\,x + C_1, \quad G(x) = -\tfrac{1}{2}\,x + C_2, \\
& F(x) + G(x) = x - 2 \quad \Rightarrow \quad C_1 + C_2 = -2 \\
\Rightarrow \quad & u(x,t) = \tfrac{3}{2}\,(x-t) + C_1 - \tfrac{1}{2}\,(x+t) + C_2 = x - 2t - 2.
\end{aligned}
$$

(ii) As in (i) but with $c = 3$,

$$
\begin{aligned}
& u(x,t) = F(x-3t) + G(x+3t) \quad \Rightarrow \quad u_t(x,t) = -3F'(x-3t) + 3G'(x+3t) \\
\Rightarrow \quad & u(x,0) = F(x) + G(x) = x^2 \quad \Rightarrow \quad F'(x) + G'(x) = 2x \\
\Rightarrow \quad & \tfrac{1}{3}\,u_x(x,0) = -F'(x) + G'(x) = \tfrac{1}{3}\,(x+1) \\
\Rightarrow \quad & F'(x) = \tfrac{5}{6}\,x - \tfrac{1}{6}, \quad G'(x) = \tfrac{7}{6}\,x + \tfrac{1}{6} \\
\Rightarrow \quad & F(x) = \tfrac{5}{12}\,x^2 - \tfrac{1}{6}\,x + C_1, \quad G(x) = \tfrac{7}{12}\,x^2 + \tfrac{1}{6}\,x + C_2, \\
& F(x) + G(x) = x^2 \quad \Rightarrow \quad C_1 + C_2 = 0 \\
\Rightarrow \quad & u(x,t) = \tfrac{5}{12}\,(x-3t)^2 - \tfrac{1}{6}\,(x-3t) + C_1 + \tfrac{7}{12}\,(x+3t)^2 + \tfrac{1}{6}\,(x+3t) + C_2 \\
& \qquad = x^2 + xt + 9t^2 + t.
\end{aligned}
$$

6. (i) By (12.6),

$$u(x,t) = F(x-t) + G(x+t).$$

If $x > t$, then

$$
\begin{aligned}
& u(x,0) = F(x) + G(x) = 2x - 1 \quad \Rightarrow \quad F'(x) + G'(x) = 2, \\
& u_t(x,0) = -F'(x) + G'(x) = 3 \quad \Rightarrow \quad F'(x) = -\tfrac{1}{2}, \quad G'(x) = \tfrac{5}{2} \\
\Rightarrow \quad & F(x) = -\tfrac{1}{2}\,x + C_1, \quad G(x) = \tfrac{5}{2}\,x + C_2, \\
& F(x) + G(x) = 2x - 1 \quad \Rightarrow \quad C_1 + C_2 = -1 \\
\Rightarrow \quad & u(x,t) = -\tfrac{1}{2}\,(x-t) + \tfrac{5}{2}\,(x+t) + C_1 + C_2 = 2x + 3t - 1.
\end{aligned}
$$

If $x < t$, then

$$\begin{aligned} & u(0,t) = F(-t) + G(t) = 0 \quad \Rightarrow \quad F(-t) = -G(t), \quad t > 0 \\ \Rightarrow \quad & F(z) = -G(z), \quad z < 0 \\ \Rightarrow \quad & u(x,t) = -G(t-x) + G(t+x) = -\tfrac{5}{2}\,(t-x) - C_2 + \tfrac{5}{2}\,(t+x) + C_2 = 5x. \end{aligned}$$

(ii) As in (i),

$$u(x,t) = F(x-t) + G(x+t).$$

If $x > \frac{1}{2}\,t$, then

$$\begin{aligned} & u(x,0) = F(x) + G(x) = \sin x \quad \Rightarrow \quad F'(x) + G'(x) = \cos x, \\ & u_t(x,0) = -F'(x) + G'(x) = 2x - 4 \\ \Rightarrow \quad & F'(x) = 2 - x + \tfrac{1}{2}\,\cos x, \quad G'(x) = x - 2 + \tfrac{1}{2}\,\cos x \\ \Rightarrow \quad & F(x) = 2x - \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,\sin x + C_1, \quad G(x) = \tfrac{1}{2}\,x^2 - 2x + \tfrac{1}{2}\,\sin x + C_2, \\ & F(x) + G(x) = \sin x \quad \Rightarrow \quad C_1 + C_2 = 0 \\ \Rightarrow \quad & u(x,t) = 2\bigl(x - \tfrac{1}{2}\,t\bigr) - \tfrac{1}{2}\,\bigl(x - \tfrac{1}{2}\,t\bigr)^2 + \tfrac{1}{2}\,\sin\bigl(x - \tfrac{1}{2}\,t\bigr) + C_1 \\ & \qquad - 2\bigl(x + \tfrac{1}{2}\,t\bigr) + \tfrac{1}{2}\,\bigl(x + \tfrac{1}{2}\,t\bigr)^2 + \tfrac{1}{2}\,\sin\bigl(x + \tfrac{1}{2}\,t\bigr) + C_2 \\ & \quad = xt - 2t + \sin x \cos\bigl(\tfrac{1}{2}\,t\bigr). \end{aligned}$$

If $x < \frac{1}{2}\,t$, then

$$\begin{aligned} u(x,t) &= -G(t-x) + G(t+x) \\ &= -\tfrac{1}{2}\,\bigl(\tfrac{1}{2}\,t - x\bigr)^2 + 2\bigl(\tfrac{1}{2}\,t - x\bigr) - \tfrac{1}{2}\,\sin\bigl(\tfrac{1}{2}\,t - x\bigr) - C_2 \\ &\qquad + \tfrac{1}{2}\,\bigl(\tfrac{1}{2}\,t + x\bigr)^2 - 2\bigl(\tfrac{1}{2}\,t + x\bigr) + \tfrac{1}{2}\,\sin\bigl(\tfrac{1}{2}\,t + x\bigr) + C_2 \\ &= xt - 4x + \sin x \cos\bigl(\tfrac{1}{2}\,t\bigr). \end{aligned}$$