2. (i) We have

$$x'(t) = \tfrac{1}{2} \quad \Rightarrow \quad x = \tfrac{1}{2}\,t + \text{const} \quad \Rightarrow \quad \frac{du}{dt} = u \quad \Rightarrow \quad u(x,t) = Ce^t.$$

If $x > \frac{1}{2}\,t$, then

$$\begin{aligned} & x(0) = x_0 \quad \Rightarrow \quad x = \tfrac{1}{2}\,t + x_0 \quad \Rightarrow \quad x_0 = x - \tfrac{1}{2}\,t \\ \Rightarrow \quad & u(x,t)e^{-t} = C = u(x_0,0) = e^{x_0} = e^{x-t/2} \\ \Rightarrow \quad & u(x,t) = e^{x+t/2}. \end{aligned}$$

If $x < \frac{1}{2}\,t$, then

$$\begin{aligned} & t(0) = t_0 \quad \Rightarrow \quad x = \tfrac{1}{2}\,t - \tfrac{1}{2}\,t_0 \quad \Rightarrow \quad t_0 = t - 2x \\ \Rightarrow \quad & u(x,t)e^{-t} = C = u(0,t_0)e^{-t_0} = e^{-t_0} = e^{2x-t} \\ \Rightarrow \quad & u(x,t) = e^{2x}. \end{aligned}$$

Since the solution is continuous across the line $x = \frac{1}{2}\,t$, we can write

$$u(x,t) = \begin{cases} e^{2x}, & x \le \tfrac{1}{2}\,t, \\ e^{x+t/2}, & x > \tfrac{1}{2}\,t. \end{cases}$$

(ii) As in (i),

$$x'(t) = 2 \quad \Rightarrow \quad x = 2t + \text{const}.$$

If $x > 2t$, then

$$\begin{aligned} & x(0) = x_0 \quad \Rightarrow \quad x = 2t + x_0 \quad \Rightarrow \quad x_0 = x - 2t \\ \Rightarrow \quad & \frac{du}{dt} = x^2 = (2t + x_0)^2 \quad \Rightarrow \quad u(x,t) = \tfrac{1}{6}\,(2t + x_0)^3 + C \\ \Rightarrow \quad & u(x,t) - \tfrac{1}{6}\,(2t + x_0)^3 = C = u(x_0,0) - \tfrac{1}{6}\,x_0^3 = x_0 - \tfrac{1}{6}\,x_0^3 \\ \Rightarrow \quad & u(x,t) = x - 2t - \tfrac{1}{6}\,(x - 2t)^3 + \tfrac{1}{6}\,x^3. \end{aligned}$$

If $x < 2t$, then

$$\begin{aligned} & t(0) = t_0 \quad \Rightarrow \quad x = 2t - 2t_0 \quad \Rightarrow \quad t_0 = \tfrac{1}{2}\,(2t - x) = t - \tfrac{1}{2}\,x \\ \Rightarrow \quad & \frac{du}{dt} = x^2 = 4(t - t_0)^2 \quad \Rightarrow \quad u(x,t) = \tfrac{4}{3}\,(t - t_0)^3 + C \\ \Rightarrow \quad & u(x,t) - \tfrac{4}{3}\,(t - t_0)^3 = C = u(0,t_0) = t_0^2 = \tfrac{1}{4}\,(2t - x)^2 \\ \Rightarrow \quad & u(x,t) = \tfrac{1}{4}\,(2t - x)^2 + \tfrac{4}{3}\,(t - t_0)^3 = \tfrac{1}{4}\,(2t - x)^2 + \tfrac{1}{6}\,x^3. \end{aligned}$$

The solution is continuous across the line $x = 2t$, so

$$u(x,t) = \begin{cases} \frac{1}{4}(2t-x)^2 + \frac{1}{6}x^3, & x \le 2t, \\ x - 2t - \frac{1}{6}(x-2t)^3 + \frac{1}{6}x^3, & x > 2t. \end{cases}$$

3. (i) With $x = x(y)$, we have

$$\begin{aligned} & x'(y) = 2, \quad x(0) = x_0 \quad \Rightarrow \quad x = 2y + x_0 \quad \Rightarrow \quad x_0 = x - 2y \\ \Rightarrow \quad & \frac{du}{dy} = u_y + u_x \frac{dx}{dy} = u \quad \Rightarrow \quad u(x,y) = Ce^y \\ \Rightarrow \quad & u(x,y)e^{-y} = C \text{ on } x = 2y + x_0. \end{aligned}$$

Let (x_1, y_1) be the point of intersection of the characteristic and the data line. Then

$$\begin{aligned} & x_1 - 2y_1 = x_0, \quad x_1 - y_1 = 1 \quad \Rightarrow \quad x_1 = 2 - x_0, \quad y_1 = 1 - x_0 \\ \Rightarrow \quad & u(x,y)e^{-y} = C = u(x_1,y_1)e^{-y_1} = (2x_1 - y_1)e^{-y_1} \\ & \qquad = \big[2(2-x_0) - (1-x_0)\big]e^{x_0-1} = (3-x_0)e^{x_0-1} \\ \Rightarrow \quad & u(x,y) = (3 - x + 2y)e^{x-2y-1}\,e^y = (3 - x + 2y)e^{x-y-1}. \end{aligned}$$

(ii) As in (i),

$$\begin{aligned} & x'(y) = -1, \quad x(0) = x_0 \quad \Rightarrow \quad x = -y + x_0 \quad \Rightarrow \quad x_0 = x + y \\ \Rightarrow \quad & \frac{du}{dy} = 2y \quad \Rightarrow \quad u(x,t) = y^2 + C, \\ & x_1 + y_1 = x_0, \quad x_1 + 2y_1 = 1 \quad \Rightarrow \quad x_1 = 2x_0 - 1, \quad y_1 = 1 - x_0 \\ \Rightarrow \quad & u(x,y) - y^2 = C = u(x_1,y_1) - y_1^2 = x_1y_1 - y_1^2 \\ & \qquad = (2x_0 - 1)(1 - x_0) - (1 - x_0)^2 = (1 - x_0)(3x_0 - 2) \\ \Rightarrow \quad & u(x,y) = (1 - x - y)(3x + 3y - 2) + y^2. \end{aligned}$$

4. (i) As in Section 12.2,

$$\begin{aligned} & x'(t) = u + t, \quad x(0) = x_0 \\ \Rightarrow \quad & \frac{du}{dt} = 1 \quad \Rightarrow \quad u(x,t) = t + C \quad \Rightarrow \quad u(x,t) - t = C = u(x_0,0) = x_0 \\ \Rightarrow \quad & u(x,t) = t + x_0 \quad \Rightarrow \quad x' = u + t = x_0 + 2t \quad \Rightarrow \quad x = x_0t + t^2 + x_0 \\ \Rightarrow \quad & u(x,t) = t + x_0 = t + \frac{x - t^2}{t+1}. \end{aligned}$$

(ii) Here

$$
\begin{aligned}
&x'(t) = 1, \quad x(0) = x_0 \quad \Rightarrow \quad x = t + x_0 \quad \Rightarrow \quad x_0 = x - t \\
\Rightarrow \quad & \frac{du}{dt} = -2tu^2 \quad \Rightarrow \quad -\frac{1}{u} = -t^2 + C \\
\Rightarrow \quad & t^2 - \frac{1}{u(x,t)} = C = -\frac{1}{u(x_0,0)} = -e^{x_0} \\
\Rightarrow \quad & u(x,t) = \frac{1}{t^2 + e^{x_0}} = \frac{1}{t^2 + e^{x-t}}.
\end{aligned}
$$