Chapter 12

1. (i) As in Section 12.1,

$$
\begin{aligned}
& x'(t) = 2, \quad x(0) = x_0 \quad \Rightarrow \quad x = 2t + x_0 \quad \Rightarrow \quad x_0 = x - 2t \\
\Rightarrow \quad & \frac{du}{dt} = t \quad \Rightarrow \quad u(x,t) = \tfrac{1}{2}\,t^2 + C \\
\Rightarrow \quad & u(x,t) - \tfrac{1}{2}\,t^2 = C = u(x_0, 0) = 1 - x_0 \\
\Rightarrow \quad & u(x,t) = \tfrac{1}{2}\,t^2 + 1 - x_0 = \tfrac{1}{2}\,t^2 + 1 - (x - 2t) = \tfrac{1}{2}\,t^2 + 2t + 1 - x.
\end{aligned}
$$

(ii) Similarly,

$$
\begin{aligned}
& x'(t) = -3t^2, \quad x(0) = x_0 \quad \Rightarrow \quad x = -t^3 + x_0 \quad \Rightarrow \quad x_0 = x + t^3 \\
\Rightarrow \quad & \frac{du}{dt} = 2 \quad \Rightarrow \quad u(x,t) = 2t + C \quad \Rightarrow \quad u(x,t) - 2t = C = u(x_0, 0) = x_0^3 \\
\Rightarrow \quad & u(x,t) = 2t + x_0^3 = 2t + (x + t^3)^3.
\end{aligned}
$$

(iii) We have

$$
\begin{aligned}
& x'(t) = 2t + 1, \quad x(0) = x_0 \quad \Rightarrow \quad x = t^2 + t + x_0 \quad \Rightarrow \quad x_0 = x - t^2 - t \\
\Rightarrow \quad & \frac{du}{dt} = 2u \quad \Rightarrow \quad u(x,t) = Ce^{2t} \\
\Rightarrow \quad & u(x,t)e^{-2t} = C = u(x_0, 0) = \sin x_0 \\
\Rightarrow \quad & u(x,t) = e^{2t} \sin x_0 = e^{2t} \sin(x - t^2 - t).
\end{aligned}
$$

(iv) Here

$$
\begin{aligned}
& x'(t) = -3, \quad x(0) = x_0 \quad \Rightarrow \quad x = -3t + x_0 \quad \Rightarrow \quad x_0 = x + 3t \\
\Rightarrow \quad & \frac{du}{dt} = 1 - x = 1 + 3t - x_0 \quad \Rightarrow \quad u(t) = t + \tfrac{3}{2}\,t^2 - x_0 t + C \\
\Rightarrow \quad & u(x,t) - t - \tfrac{3}{2}\,t^2 + x_0 t = C = u(x_0, 0) = x_0^2 + 1 \\
\Rightarrow \quad & u(x,t) = t + \tfrac{3}{2}\,t^2 - x_0 t + x_0^2 + 1 = t + \tfrac{3}{2}\,t^2 - t(x + 3t) + (x + 3t)^2 + 1 \\
& \qquad\quad = \tfrac{15}{2}\,t^2 + t + 1 + 5xt + x^2.
\end{aligned}
$$