Chapter 11

1. (i) Here

$$\begin{aligned} &A=1,\ B=x-1,\ C=1 \quad\Rightarrow\quad B^2-4AC=(x-1)^2-4\\ \Rightarrow\quad &(x-1)^2<4 \quad\Leftrightarrow\quad |x-1|<2 \quad\Leftrightarrow\quad -1<x<3: \quad \text{elliptic},\\ &x<-1 \text{ or } x>3: \quad \text{hyperbolic},\\ &x=-1 \text{ or } x=3: \quad \text{parabolic}. \end{aligned}$$

(ii) Similarly,

$$\begin{aligned} &A=y,\ B=-x,\ C=y \quad\Rightarrow\quad B^2-4AC=x^2-4y^2=(x-2y)(x+2y)\\ \Rightarrow\quad &|x|<2|y| \text{ (obtuse angles between the lines } x=2y \text{ and } x=-2y\text{): elliptic},\\ &|x|>2|y| \text{ (acute angles between the lines } x=2y \text{ and } x=-2y\text{): hyperbolic},\\ &|x|=2|y| \text{ (on the lines } x=2y \text{ and } y=-2y\text{): parabolic}. \end{aligned}$$

(iii) This time

$$\begin{aligned} &A=x+2,\ B=2(x+y),\ C=2(y-1)\\ \Rightarrow\quad &B^2-4AC=4(x+y)^2-8(x+2)(y-1)=4(x^2+y^2+2x-4y+4)\\ &\qquad\qquad\;=4\big[(x+1)^2+(y-2)^2-1\big]\\ \Rightarrow\quad &\text{elliptic inside the circle } (x+1)^2+(y-2)^2=1,\\ &\text{hyperbolic outside the circle } (x+1)^2+(y-2)^2=1,\\ &\text{parabolic on the circle } (x+1)^2+(y-2)^2=1. \end{aligned}$$

2. (i) We have

$$\begin{aligned} &B^2-4AC=49+32=81>0 \quad\Rightarrow\quad \text{hyperbolic equation},\\ &\frac{dy}{dx}=\frac{-7\pm 9}{8}=-2,\ \tfrac{1}{4} \quad\Rightarrow\quad y=-2x+C_1, \quad y=\tfrac{1}{4}\,x+C_2\\ \Rightarrow\quad &r=2x+y, \quad s=4y-x\\ \Rightarrow\quad &\bar{A}=\bar{C}=0,\ \bar{B}=-81,\ \bar{D}=\bar{E}=\bar{F}=0,\ \bar{G}=243rs^2, \quad u(x,y)=v(r,s)\\ \Rightarrow\quad &v_{rs}=-3rs^2 \quad\Rightarrow\quad v_r=-rs^3+C_1(r)\\ \Rightarrow\quad &v(r,s)=-\tfrac{1}{2}\,r^2s^3+\varphi(r)+\psi(s)\\ \Rightarrow\quad &u(x,y)=\tfrac{1}{2}\,(2x+y)^2(x-4y)^3+\varphi(2x+y)+\psi(4y-x). \end{aligned}$$

(ii) Here

$$B^2 - 4AC = 1 + 24 = 25 > 0 \quad \Rightarrow \quad \text{hyperbolic equation,}$$

$$\frac{dy}{dx} = \frac{1 \pm 5}{6} = 1, -\tfrac{2}{3} \quad \Rightarrow \quad y = x + C_1, \quad y = -\tfrac{2}{3}\,x + C_2$$

$$\Rightarrow \quad r = y - x, \quad s = 2x + 3y$$

$$\Rightarrow \quad \bar{A} = \bar{C} = 0,\ \bar{B} = -25,\ \bar{D} = 50,\ \bar{E} = \bar{F} = 0,\ \bar{G} = 25s, \quad u(x,y) = v(r,s)$$

$$\Rightarrow \quad v_{rs} - 2v_r = -s$$

$$\Rightarrow \quad v_r = e^{2s} \int -se^{-2s}\, ds = e^{2s}\left(\tfrac{1}{2}\, se^{-2s} - \int \tfrac{1}{2}\, e^{-2s}\, ds\right) = \tfrac{1}{2}\, s + \tfrac{1}{4} + C_1(r)e^{2s}$$

$$\Rightarrow \quad v(r,s) = \tfrac{1}{2}\, rs + \tfrac{1}{4}\, r + \varphi(r)e^{2s} + \psi(s)$$

$$\Rightarrow \quad u(x,y) = \tfrac{1}{2}\,(2x+3y)(y-x) + \tfrac{1}{4}\,(y-x) + \varphi(y-x)e^{2(2x+3y)} + \psi(2x+3y).$$