5. (i) We have

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad X_n(x) = \sin\frac{(2n-1)\pi x}{2}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{(2n-1)\pi x}{2},$$

$$q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin\frac{(2n-1)\pi x}{2}, \quad q_n(t) = 2\int_0^1 q(\xi,t) \sin\frac{(2n-1)\pi\xi}{2}\, d\xi,$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\frac{(2n-1)\pi x}{2}, \quad f_n = 2\int_0^1 f(\xi) \sin\frac{(2n-1)\pi x}{2}\, d\xi$$

$$\Rightarrow \quad u_n'(t) + \frac{(2n-1)^2\pi^2}{4} u_n(t) = q_n(t), \quad t > 0, \quad u_n(0) = f_n$$

$$\Rightarrow \quad u_n(t) = e^{-(2n-1)^2\pi^2 t/4}\left[\int_0^t q_n(\tau) e^{(2n-1)^2\pi^2\tau/4}\, d\tau + C\right]$$

$$= f_n e^{-(2n-1)^2\pi^2 t/4} + e^{-(2n-1)^2\pi^2 t/4}\int_0^t q_n(\tau) e^{(2n-1)^2\pi^2\tau/4}\, d\tau, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty}\Bigg[f_n e^{-(2n-1)^2\pi^2 t/4} + e^{-(2n-1)^2\pi^2 t/4}\int_0^t q_n(\tau) e^{(2n-1)^2\pi^2\tau/4}\, d\tau\Bigg] \sin\frac{(2n-1)\pi x}{2}$$

$$= \sum_{n=1}^{\infty}\Bigg\{\left[2\int_0^1 f(\xi)\sin\frac{(2n-1)\pi\xi}{2}\, d\xi\right] e^{-(2n-1)^2\pi^2 t/4} + e^{-(2n-1)^2\pi^2 t/4}\int_0^t \left[2\int_0^1 q(\xi,\tau)\sin\frac{(2n-1)\pi\xi}{2}\, d\xi\right] e^{(2n-1)^2\pi^2\tau/4}\, d\tau\Bigg\} \times \sin\frac{(2n-1)\pi x}{2}$$

$$= \int_0^1 f(\xi) \Big[\sum_{n=1}^{\infty} 2 \sin \frac{(2n-1)\pi x}{2} \, \sin \frac{(2n-1)\pi \xi}{2} \, e^{-(2n-1)^2 \pi^2 t/4} \Big] \, d\xi$$

$$+ \int_0^1 \int_0^t q(\xi, \tau) \Big[\sum_{n=1}^{\infty} 2 \sin \frac{(2n-1)\pi x}{2} \, \sin \frac{(2n-1)\pi \xi}{2} \times e^{-(2n-1)^2 \pi^2 (t-\tau)/4} \Big] \, d\tau \, d\xi$$

$$\Rightarrow \quad G(x, t; \xi, \tau) = \sum_{n=1}^{\infty} 2 \sin \frac{(2n-1)\pi x}{2} \, \sin \frac{(2n-1)\pi \xi}{2} \, e^{-(2n-1)^2 \pi^2 (t-\tau)/4}, \quad \tau < t$$

$$\Rightarrow \quad u(x, t) = \int_0^1 G(x, t; \xi, 0) f(\xi) \, d\xi + \int_0^1 \int_0^t G(x, t; \xi, \tau) q(\xi, \tau) \, d\tau \, d\xi,$$

$$\int_0^1 f(\xi) \sin \frac{(2n-1)\pi \xi}{2} \, d\xi = \int_0^1 \xi \sin \frac{(2n-1)\pi \xi}{2} \, d\xi$$

$$= \Big[\xi \Big(-\frac{2}{(2n-1)\pi} \Big) \cos \frac{(2n-1)\pi \xi}{2} \Big]_0^1 + \int_0^1 \frac{2}{(2n-1)\pi)} \, \cos \frac{(2n-1)\pi \xi}{2} \, d\xi$$

$$= \frac{4}{(2n-1)^2 \pi^2} \Big[\sin \frac{(2n-1)\pi \xi}{2} \Big]_0^1 = (-1)^{n+1} \frac{4}{(2n-1)^2 \pi^2}, \quad n = 1, 2, \ldots,$$

$$\int_0^1 \int_0^t q(\xi, \tau) \sin \frac{(2n-1)\pi \xi}{2} \, e^{(2n-1)^2 \pi^2 \tau/4} \, d\tau \, d\xi$$

$$= \Big[\int_0^1 \sin \frac{(2n-1)\pi \xi}{2} \, d\xi \Big] \Big[\int_0^t e^{(2n-1)^2 \pi^2 \tau/4} \, d\tau \Big]$$

$$= -\frac{2}{(2n-1)\pi)} \Big[\cos \frac{(2n-1)\pi \xi}{2} \Big]_0^1 \frac{4}{(2n-1)^2 \pi^2} \big[e^{(2n-1)^2 \pi^2 \tau/4} \big]_0^t$$

$$= \frac{8}{(2n-1)^3 \pi^3} \big[e^{(2n-1)^2 \pi^2 t/4} - 1 \big], \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x, t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{(2n-1)^2 \pi^2} \, 2 \sin \frac{(2n-1)\pi x}{2} \, e^{-(2n-1)^2 \pi^2 t/4}$$

$$+\sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi^3} \left[e^{(2n-1)^2\pi^2 t/4} - 1\right] 2\sin\frac{(2n-1)\pi x}{2} \times e^{-(2n-1)^2\pi^2 t/4}$$

$$= \sum_{n=1}^{\infty} \left\{ (-1)^{n+1} \frac{8}{(2n-1)^2\pi^2} e^{-(2n-1)^2\pi^2 t/4} + \frac{16}{(2n-1)^3\pi^3} \left[1 - e^{-(2n-1)^2\pi^2 t/4}\right] \right\} \sin\frac{(2n-1)\pi x}{2}.$$

(ii) With G as in (i), we have

$$\int_0^1 f(\xi)\sin\frac{(2n-1)\pi\xi}{2}\,d\xi = \int_0^1 \sin\frac{(2n-1)\pi\xi}{2}\,d\xi = \frac{2}{(2n-1)\pi}, \quad n = 1, 2, \ldots,$$

$$\int_0^1\int_0^t q(\xi,\tau)\sin\frac{(2n-1)\pi\xi}{2} e^{(2n-1)^2\pi^2\tau/4}\,d\tau\,d\xi$$

$$= \left[\int_0^1 \sin\frac{(2n-1)\pi\xi}{2}\,d\xi\right]\left[\int_0^t \tau e^{(2n-1)^2\pi^2\tau/4}\,d\tau\right]$$

$$= \frac{2}{(2n-1)\pi)} \left\{ \tau \frac{4}{(2n-1)^2\pi^2} \left[e^{(2n-1)^2\pi^2\tau/4}\right]_0^t - \int_0^t \frac{4}{(2n-1)^2\pi^2} e^{(2n-1)^2\pi^2\tau/4}\,d\tau \right\}$$

$$= \frac{2}{(2n-1)\pi} \left\{ \frac{4}{(2n-1)^2\pi^2} te^{(2n-1)^2\pi^2 t/4} - \frac{16}{(2n-1)^4\pi^4} \left[e^{(2n-1)^2\pi^2\tau/4}\right]_0^t \right\}$$

$$= \frac{8}{(2n-1)^3\pi^3} te^{(2n-1)^2\pi^2 t/4} - \frac{32}{(2n-1)^5\pi^5} \left[e^{(2n-1)^2\pi^2 t/4} - 1\right], \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} 2\sin\frac{(2n-1)\pi x}{2} e^{-(2n-1)^2\pi^2 t/4}$$

$$+\sum_{n=1}^{\infty}\left\{\frac{8}{(2n-1)^3\pi^3}\,te^{(2n-1)^2\pi^2t/4}-\frac{32}{(2n-1)^5\pi^5}\left[e^{(2n-1)^2\pi^2t/4}-1\right]\right\}$$

$$\times\, 2\sin\frac{(2n-1)\pi x}{2}\,e^{-(2n-1)^2\pi^2t/4}$$

$$=\sum_{n=1}^{\infty}\left\{\frac{4}{(2n-1)\pi}\,e^{-(2n-1)^2\pi^2t/4}+\frac{16}{(2n-1)^3\pi^3}\,t\right.$$

$$\left.-\frac{64}{(2n-1)^5\pi^5}\left[1-e^{-(2n-1)^2\pi^2t/4}\right]\right\}\sin\frac{(2n-1)\pi x}{2}.$$

6. (i) We have

$$\lambda_n=\frac{(2n-1)^2\pi^2}{4},\quad X_n(x)=\cos\frac{(2n-1)\pi x}{2},\quad n=1,2,\ldots$$

$$\Rightarrow\quad u(x,t)=\int_0^1 f(\xi)\left[\sum_{n=1}^{\infty}2\cos\frac{(2n-1)\pi x}{2}\,\cos\frac{(2n-1)\pi\xi}{2}\,e^{-(2n-1)^2\pi^2t/4}\right]d\xi$$

$$+\int_0^1\int_0^t q(\xi,\tau)\left[\sum_{n=1}^{\infty}2\cos\frac{(2n-1)\pi x}{2}\,\cos\frac{(2n-1)\pi\xi}{2}\,e^{-(2n-1)^2\pi^2(t-\tau)/4}\right]d\tau\,d\xi$$

$$\Rightarrow\quad G(x,t;\xi,\tau)=\sum_{n=1}^{\infty}2\cos\frac{(2n-1)\pi x}{2}\,\cos\frac{(2n-1)\pi\xi}{2}\,e^{-(2n-1)^2\pi^2(t-\tau)/4},\quad \tau<t$$

$$\Rightarrow\quad u(x,t)=\int_0^1 G(x,t;\xi,0)f(\xi)\,d\xi+\int_0^1\int_0^t G(x,t;\xi,\tau)q(\xi,\tau)\,d\tau\,d\xi,$$

$$\int_0^1 f(\xi)\cos\frac{(2n-1)\pi\xi}{2}\,d\xi=\int_0^1 \xi\cos\frac{(2n-1)\pi\xi}{2}\,d\xi$$

$$=\left[\xi\,\frac{2}{(2n-1)\pi}\,\sin\frac{(2n-1)\pi\xi}{2}\right]_0^1-\int_0^1\frac{2}{(2n-1)\pi}\,\sin\frac{(2n-1)\pi\xi}{2}\,d\xi$$

$$=\frac{2}{(2n-1)\pi}\,\sin\frac{(2n-1)\pi}{2}+\frac{4}{(2n-1)^2\pi^2}\left[\cos\frac{(2n-1)\pi\xi}{2}\right]_0^1$$

$$=(-1)^{n+1}\frac{2}{(2n-1)\pi}-\frac{4}{(2n-1)^2\pi^2},\quad n=1,2,\ldots,$$

$$\int_0^1 \int_0^t q(\xi,\tau) \cos \frac{(2n-1)\pi\xi}{2}\, e^{(2n-1)^2\pi^2\tau/4}\, d\tau\, d\xi$$

$$= \left[\int_0^1 \cos \frac{(2n-1)\pi\xi}{2}\, d\xi\right] \left[\int_0^t e^{(2n-1)^2\pi^2\tau/4}\, d\tau\right]$$

$$= \frac{2}{(2n-1)\pi} \left[\sin \frac{(2n-1)\pi\xi}{2}\right]_0^1 \frac{4}{(2n-1)^2\pi^2} \left[e^{(2n-1)^2\pi^2\tau/4}\right]_0^t$$

$$= (-1)^{n+1} \frac{8}{(2n-1)^3\pi^3} \left[e^{(2n-1)^2\pi^2 t/4} - 1\right], \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \left[(-1)^{n+1} \frac{2}{(2n-1)\pi} - \frac{4}{(2n-1)^2\pi^2}\right] 2 \cos \frac{(2n-1)\pi x}{2}\, e^{-(2n-1)^2\pi^2 t/4}$$

$$+ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{8}{(2n-1)^3\pi^3} \left[e^{(2n-1)^2\pi^2 t/4} - 1\right]$$

$$\times 2 \cos \frac{(2n-1)\pi x}{2}\, e^{-(2n-1)^2\pi^2 t/4}$$

$$= \sum_{n=1}^{\infty} \left\{ (-1)^{n+1} \frac{16}{(2n-1)^3\pi^3} + \left[(-1)^{n+1} \frac{4}{(2n-1)\pi} - \frac{8}{(2n-1)^2\pi^2} \right.\right.$$

$$\left.\left. + (-1)^n \frac{16}{(2n-1)^3\pi^3}\right] e^{-(2n-1)^2\pi^2 t/4} \right\} \cos \frac{(2n-1)\pi x}{2}.$$

(ii) With G as in (i),

$$\int_0^1 f(\xi) \cos \frac{(2n-1)\pi\xi}{2}\, d\xi = \int_0^1 \cos \frac{(2n-1)\pi\xi}{2}\, d\xi$$

$$= \frac{2}{(2n-1)\pi} \left[\sin \frac{(2n-1)\pi\xi}{2}\right]_0^1 = (-1)^{n+1} \frac{2}{(2n-1)\pi}, \quad n = 1, 2, \ldots,$$

$$\int_0^1 \int_0^t q(\xi,\tau) \cos \frac{(2n-1)\pi\xi}{2}\, e^{(2n-1)^2\pi^2\tau/4}\, d\tau\, d\xi$$

$$= \left[\int_0^1 \cos \frac{(2n-1)\pi\xi}{2}\, d\xi\right] \left[\int_0^t \tau e^{(2n-1)^2\pi^2\tau/4}\, d\tau\right]$$

$$= \frac{2}{(2n-1)\pi}\left[\sin\frac{(2n-1)\pi\xi}{2}\right]_0^1$$

$$\times\left\{\frac{4}{(2n-1)^2\pi^2}\,te^{(2n-1)^2\pi^2t/4} - \frac{16}{(2n-1)^4\pi^4}\left[e^{(2n-1)^2\pi^2t/4}-1\right]\right\}$$

$$= (-1)^{n+1}\frac{8}{(2n-1)^3\pi^3}\,te^{(2n-1)^2\pi^2t/4}$$

$$+ (-1)^n\frac{32}{(2n-1)^5\pi^5}\left[e^{(2n-1)^2\pi^2t/4}-1\right], \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{(2n-1)\pi}\,2\cos\frac{(2n-1)\pi x}{2}\,e^{-(2n-1)^2\pi^2t/4}$$

$$+ \sum_{n=1}^{\infty}\left\{(-1)^{n+1}\frac{8}{(2n-1)^3\pi^3}\,te^{(2n-1)^2\pi^2t/4}\right.$$

$$\left. + (-1)^n\frac{32}{(2n-1)^5\pi^5}\left[e^{(2n-1)^2\pi^2t/4}-1\right]\right\}$$

$$\times 2\cos\frac{(2n-1)\pi x}{2}\,e^{-(2n-1)^2\pi^2t/4}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\left\{\frac{16}{(2n-1)^3\pi^3}\,t - \frac{64}{(2n-1)^5\pi^5}\right.$$

$$\left. + \left[\frac{4}{(2n-1)\pi} + \frac{64}{(2n-1)^5\pi^5}\right]e^{-(2n-1)^2\pi^2t/4}\right\}$$

$$\times \cos\frac{(2n-1)\pi x}{2}.$$