Chapter 10

1. (i) As in Section 10.1,

$$G_{xx}(x,\xi) = -\delta(x-\xi), \quad 0<x<1, \quad G(0,\xi)=0, \quad G_x(1,\xi)=0$$

$$\Rightarrow \quad G_x(x,\xi) = -H(x-\xi) + C_1(\xi) = \begin{cases} C_1(\xi), & x<\xi, \\ -1+C_1(\xi), & x>\xi, \end{cases}$$

$$\Rightarrow \quad G(x,\xi) = \begin{cases} xC_1(\xi)+C_2(\xi), & x<\xi, \\ -x+xC_1(\xi)+C_3(\xi), & x>\xi \end{cases}$$

$$\Rightarrow \quad C_2(\xi)=0, \quad -1+C_1(\xi)=0 \quad \Rightarrow \quad G(x,\xi) = \begin{cases} x, & x<\xi, \\ C_3(\xi), & x>\xi, \end{cases}$$

$$G(\xi-,\xi) = G(\xi+,\xi) \quad \Rightarrow \quad C_3(\xi)=\xi \quad \Rightarrow \quad G(x,\xi) = \begin{cases} x, & x\le\xi, \\ \xi, & x>\xi \end{cases}$$

$$\Rightarrow \quad G(x,\xi) = G(\xi,x), \quad G_\xi(x,\xi) = \begin{cases} 0, & x<\xi, \\ 1, & x>\xi. \end{cases}$$

If in Green's formula

$$\int_0^1 (u''v - v''u)\,dx = \big[u'v - v'u\big]_0^1$$

we let u be the solution of the given BVP and $v=G$, then

$$\int_0^1 \big[-G(x,\xi) + \delta(x-\xi)u(x)\big]\,dx = -G(1,\xi) + G_x(0,\xi)$$

$$\Rightarrow \quad u(\xi) = \int_0^1 G(x,\xi)\,dx - G(1,\xi) + G_x(0,\xi)$$

$$\Rightarrow \quad u(x) = \int_0^1 G(x,\xi)\,d\xi - G(x,1) + G_\xi(x,0)$$

$$= \int_0^x \xi\,d\xi + \int_x^1 x\,d\xi - x + 1 = \tfrac{1}{2}x^2 + x(1-x) - x + 1 = -\tfrac{1}{2}x^2 + 1.$$

(ii) The Green's function is the same as in (i), but this time Green's formula yields

$$\int_0^1 \big[-xG(x,\xi)+\delta(x-\xi)u(x)\big]\,dx = G(1,\xi) - 2G_x(0,\xi)$$

$$\begin{aligned}\Rightarrow\quad u(x) &= \int_0^1 \xi G(x,\xi)\,d\xi + G(x,1) - 2G_\xi(x,0)\\ &= \int_0^x \xi^2\,\delta\xi + \int_x^1 \xi x\,d\xi + x - 2\\ &= \tfrac{1}{3}x^3 + \tfrac{1}{2}x(1-x^2) + x - 2 = -\tfrac{1}{6}x^3 + \tfrac{3}{2}x - 2.\end{aligned}$$

2. (i) Here

$$G_{xx}(x,\xi) = -\delta(x-\xi),\quad 0<x<1,\quad G_x(0,\xi)=0,\quad G(1,\xi)=0$$

$$\Rightarrow\quad G_x(x,\xi) = -H(x-\xi) + C_1(\xi) = \begin{cases} C_1(\xi), & x<\xi,\\ -1+C_1(\xi), & x>\xi\end{cases}$$

$$\Rightarrow\quad G(x,\xi) = \begin{cases} xC_1(\xi)+C_2(\xi), & x<\xi,\\ -x+xC_1(\xi)+C_3(\xi), & x>\xi\end{cases}$$

$$\Rightarrow\quad C_1(\xi)=0,\quad -1+C_3(\xi)=0 \quad\Rightarrow\quad G(x,\xi) = \begin{cases} C_2(\xi), & x<\xi,\\ -x+1, & x>\xi,\end{cases}$$

$$G(\xi-,\xi) = G(\xi+,\xi) \quad\Rightarrow\quad C_2(\xi) = -\xi+1 \quad\Rightarrow\quad G(x,\xi) = \begin{cases} -\xi+1, & x\le\xi,\\ -x+1, & x>\xi\end{cases}$$

$$\Rightarrow\quad G(x,\xi) = G(\xi,x),\quad G_\xi(x,\xi) = \begin{cases} -1, & x<\xi,\\ 0, & x>\xi.\end{cases}$$

Green's formula (10.5) with u the solution of the given BVP and v replaced by G now yields

$$\int_0^1 \big[-G(x,\xi)+\delta(x-\xi)u(x)\big]\,dx = G_x(1,\xi) - G(0,\xi)$$

$$\Rightarrow\quad u(\xi) = \int_0^1 G(x,\xi)\,dx + G_x(1,\xi) - G(0,\xi)$$

$$\Rightarrow \quad u(x) = \int_0^1 G(x,\xi)\,d\xi + G_\xi(x,1) - G(x,0)$$

$$= \int_0^x (-x+1)\,d\xi + \int_x^1 (-\xi+1)\,d\xi - 1 + x - 1$$

$$= (-x+1)x - \tfrac{1}{2}\,(1-x^2) + 1 - x + x - 2 = -\tfrac{1}{2}\,x^2 + x - \tfrac{3}{2}.$$

(ii) Using the function G from (i), Green's formula (10.5), and the BCs of the given BVP, we obtain

$$\int_0^1 \big[-xG(x,\xi) + \delta(x-\xi)u(x)\big]\,dx = -G_x(1,\xi) + 2G(0,\xi)$$

$$\Rightarrow \quad u(x) = \int_0^1 \xi G(x,\xi)\,d\xi - G_\xi(x,1) + 2G(x,0)$$

$$= \int_0^x \xi(-x+1)\,d\xi + \int_x^1 \xi(-\xi+1)\,d\xi + 1 - 2x + 2$$

$$= (-x+1)\,\tfrac{1}{2}\,x^2 - \tfrac{1}{3}\,(1-x^3) + \tfrac{1}{2}\,(1-x^2) - 2x + 3 = -\tfrac{1}{6}\,x^3 - 2x + \tfrac{19}{6}.$$

3. The eigenvalue-eigenvector pairs of the BVP are (see Example 5.5)

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad X_n(x) = \sin\frac{(2n-1)\pi x}{2}, \quad n = 1, 2, \dots$$

In view of the continuity of G at $x = \xi$ and the symmetry $G(x,\xi) = G(\xi,x)$, we seek a representation of the form

$$G(x,\xi) = \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} b_{mn}\sin\frac{(2n-1)\pi x}{2}\right]\sin\frac{(2m-1)\pi\xi}{2}.$$

Substituting in the ODE for G, we find that

$$-\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{(2n-1)^2\pi^2}{4}\,b_{mn}\sin\frac{(2n-1)\pi x}{2}\right]\sin\frac{(2m-1)\pi\xi}{2} = -\delta(x-\xi).$$

We multiply both sides by $\sin\big((2p-1)\pi x/2\big)$ and integrate over $[0,1]$. Since (see Example 5.5) the set $\{X_n\}_{n=1}^{\infty}$ is orthogonal on $[0,1]$ and

$$\int_0^1 \sin^2\frac{(2p-1)\pi x}{2}\,dx = \frac{1}{2}, \quad n = 1, 2, \ldots,$$

we obtain

$$\frac{(2p-1)^2\pi^2}{4}\,\frac{1}{2}\sum_{n=1}^{\infty} b_{mp}\sin\frac{(2m-1)\pi\xi}{2} = \sin\frac{(2p-1)\pi\xi}{2}$$

$$\Rightarrow \quad b_{pp} = \frac{8}{(2p-1)^2\pi^2}, \quad b_{mp} = 0,\ m, p = 1, 2, \ldots,\ m \neq p$$

$$\Rightarrow \quad G(x,\xi) = \sum_{n=1}^{\infty}\frac{8}{(2n-1)^2\pi^2}\,\sin\frac{(2n-1)\pi x}{2}\,\sin\frac{(2n-1)\pi\xi}{2}.$$

4. Here (see Example 5.6)

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad X_n(x) = \cos\frac{(2n-1)\pi x}{2}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad G(x,\xi) = \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} a_{mn}\cos\frac{(2n-1)\pi x}{2}\right]\cos\frac{(2m-1)\pi\xi}{2}$$

$$\Rightarrow \quad \frac{(2p-1)^2\pi^2}{4}\,\frac{1}{2}\sum_{m=1}^{\infty} a_{mp}\cos\frac{(2m-1)\pi\xi}{2} = \cos\frac{(2p-1)\pi\xi}{2}$$

$$\Rightarrow \quad a_{pp} = \frac{8}{(2p-1)^2\pi^2}, \quad a_{mp} = 0,\ m, p = 1, 2, \ldots,\ m \neq p$$

$$\Rightarrow \quad G(x,\xi) = \sum_{n=1}^{\infty}\frac{8}{(2n-1)^2\pi^2}\,\cos\frac{(2n-1)\pi x}{2}\,\cos\frac{(2n-1)\pi\xi}{2}.$$