Chapter 2

1. (i) Here $L = 1$, so, by (2.2) and (2.7)–(2.9),

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_{-1}^{0} dx = \frac{1}{2},$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0}\cos(n\pi x)\,dx = \frac{1}{n\pi}\big[\sin(n\pi x)\big]_{-1}^{0} = 0,$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{0}\sin(n\pi x)\,dx = -\frac{1}{n\pi}\big[\cos(n\pi x)\big]_{-1}^{0}$$

$$= -\frac{1}{n\pi}\big[1-\cos(n\pi)\big] = \big[(-1)^n - 1\big]\,\frac{1}{n\pi}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty}\big[(-1)^n - 1\big]\,\frac{1}{n\pi}\,\sin(n\pi x),$$

$$(\text{series}) = \begin{cases} 1, & -1 < x < 0, \\ 0, & 0 < x < 1, \\ \frac{1}{2}, & x = -1, 0, 1. \end{cases}$$

(ii) Similarly, with $L = 2$

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x) = \frac{1}{4}\left\{\int_{-2}^{0} -\,dx + \int_{0}^{2}(2-x)\,dx\right\}$$

$$= \tfrac{1}{4}\big\{-2 + \big[2x - \tfrac{1}{2}\,x^2\big]_0^2\big\} = 0,$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\left\{\int_{-2}^{0} -\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2}(2-x)\cos\frac{n\pi x}{2}\,dx\right\}$$

$$= \frac{1}{2}\left\{-\frac{2}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_{-2}^{0} + \left[(2-x)\,\frac{2}{n\pi}\,\sin\frac{n\pi x}{2}\right]_0^2 - \int_{0}^{2}\frac{2}{n\pi}\,\sin\frac{n\pi x}{2}\,(-dx)\right\}$$

$$= \frac{1}{2}\left\{ -\frac{4}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_0^2 \right\} = -\frac{2}{n^2\pi^2}\left[\cos(n\pi) - 1\right]$$

$$= \left[1 - (-1)^n\right]\frac{2}{n^2\pi^2}, \quad n = 1, 2, \ldots,$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\left\{ \int_{-2}^{0} -\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2}(2-x)\sin\frac{n\pi x}{2}\,dx \right\}$$

$$= \frac{1}{2}\left\{ \frac{2}{n\pi}\left[\cos\frac{n\pi x}{2}\right]_{-2}^{0} \right.$$

$$\left. + \left[(2-x)\left(-\frac{2}{n\pi}\right)\cos\frac{n\pi x}{2}\right]_0^2 + \int_0^2 \frac{2}{n\pi}\cos\frac{n\pi x}{2}\,(-dx) \right\}$$

$$= \frac{1}{2}\left\{ \frac{2}{n\pi}\left[1 - \cos(n\pi)\right] + \frac{4}{n\pi} - \frac{4}{n^2\pi^2}\left[\sin\frac{n\pi x}{2}\right]_0^2 \right\}$$

$$= \left[3 - (-1)^n\right]\frac{1}{n\pi}, \quad n = 1, 2, \ldots,$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty}\left\{ \left[1 - (-1)^n\right]\frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2} + \left[3 - (-1)^n\right]\frac{1}{n\pi}\sin\frac{n\pi x}{2} \right\},$$

$$(\text{series}) = \begin{cases} -1, & -2 < x < 0, \\ \frac{1}{2}, & x = 0, \\ 2 - x, & 0 < x < 2, \\ -\frac{1}{2}, & x = -2, 2. \end{cases}$$

(iii) With $L = 1$,

$$a_0 = \tfrac{1}{2}\int_{-1}^{1} f(x)\,dx = \tfrac{1}{2}\left\{ \int_{-1}^{0} x^2\,dx + \int_0^1 (1+2x)\,dx \right\}$$

$$= \tfrac{1}{2}\left\{ \tfrac{1}{3}\left[x^3\right]_{-1}^{0} + \left[x + x^2\right]_0^1 \right\} = \tfrac{7}{6},$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} x^2\cos(n\pi x)\,dx + \int_0^1 (1+2x)\cos(n\pi x)\,dx$$

$$= \left[x^2 \frac{1}{n\pi} \sin(n\pi x)\right]_{-1}^{0} - \int_{-1}^{0} \frac{1}{n\pi} \sin(n\pi x) \cdot 2x \, dx$$

$$+ \left[(1+2x) \frac{1}{n\pi} \sin(n\pi x)\right]_{0}^{1} - \int_{0}^{1} \frac{1}{n\pi} \sin(n\pi x) \cdot 2 \, dx$$

$$= -\frac{2}{n\pi}\left\{ \left[x\left(-\frac{1}{n\pi}\right) \cos(n\pi x)\right]_{-1}^{0} + \int_{-1}^{0} \frac{1}{n\pi} \cos(n\pi x)\, dx \right\}$$

$$+ \frac{2}{n^2\pi^2} \left[\cos(n\pi x)\right]_0^1$$

$$= -\frac{2}{n\pi}\left\{ -\frac{1}{n\pi} \cos(n\pi) + \frac{1}{n^2\pi^2} \left[\sin(n\pi x)\right]_{-1}^{0} \right\} + \frac{2}{n^2\pi^2} \left[\cos(n\pi) - 1\right]$$

$$= \frac{2}{n^2\pi^2} \cos(n\pi) + \frac{2}{n^2\pi^2} \cos(n\pi) - \frac{2}{n^2\pi^2} = (-1)^n \frac{4}{n^2\pi^2} - \frac{2}{n^2\pi^2}, \quad n = 1, 2, \ldots,$$

$$b_n = \int_{-1}^{1} f(x) \sin(n\pi x)\, dx = \int_{-1}^{0} x^2 \sin(n\pi x)\, dx + \int_{0}^{1} (1+2x) \sin(n\pi x)\, dx$$

$$= \left[x^2\left(-\frac{1}{n\pi}\right) \cos(n\pi x)\right]_{-1}^{0} + \int_{-1}^{0} \frac{1}{n\pi} \cos(n\pi x) \cdot 2x \, dx$$

$$+ \left[(1+2x)\left(-\frac{1}{n\pi}\right) \cos(n\pi x)\right]_{0}^{1} + \int_{0}^{1} \frac{1}{n\pi} \cos(n\pi x) \cdot 2 \, dx$$

$$= \frac{1}{n\pi} \cos(n\pi) + \frac{2}{n\pi} \left\{ \left[x \frac{1}{n\pi} \sin(n\pi x)\right]_{-1}^{0} - \int_{-1}^{0} \frac{1}{n\pi} \sin(n\pi x)\, dx \right\}$$

$$- \frac{3}{n\pi} \cos(n\pi) + \frac{1}{n\pi} + \frac{2}{n^2\pi^2} \left[\sin(n\pi x)\right]_0^1$$

$$= \frac{1}{n\pi} \cos(n\pi) + \frac{2}{n^3\pi^3} \left[\cos(n\pi x)\right]_{-1}^{0} - \frac{3}{n\pi} \cos(n\pi) + \frac{1}{n\pi}$$

$$= -\frac{2}{n\pi} \cos(n\pi) + \frac{1}{n\pi} + \frac{2}{n^3\pi^3} \left[1 - \cos(n\pi)\right]$$

$$= \left[1 - (-1)^n 2\right] \frac{1}{n\pi} + \left[1 - (-1)^n\right] \frac{2}{n^3\pi^3}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{7}{6} + \sum_{n=1}^{\infty} \left\{ \left[(-1)^n \frac{4}{n^2\pi^2} - \frac{2}{n^2\pi^2} \right] \cos(n\pi x) \right.$$
$$\left. + \left[\frac{1-(-1)^n 2}{n\pi} + 2\, \frac{1-(-1)^n}{n^3\pi^3} \right] \sin(n\pi x) \right\},$$

$$\text{(series)} = \begin{cases} x^2, & -1 < x < 0, \\ \frac{1}{2}, & x = 0, \\ 1+2x, & 0 < x < 1, \\ 2, & x = -1,\, 1. \end{cases}$$

(iv) Similarly, but with $L = 2$,

$$a_0 = \frac{1}{4} \int_{-2}^{2} f(x)\, dx = \frac{1}{4} \int_{-1}^{2} (1+x)\, dx = \tfrac{1}{4} \left[x + \tfrac{1}{2}\, x^2 \right]_{-1}^{2} = \tfrac{9}{8},$$

$$a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n\pi x}{2}\, dx = \frac{1}{2} \int_{-1}^{2} (1+x) \cos \frac{n\pi x}{2}\, dx$$
$$= \frac{1}{2} \left\{ \left[(1+x)\, \frac{2}{n\pi}\, \sin \frac{n\pi x}{2} \right]_{-1}^{2} - \int_{-1}^{2} \frac{2}{n\pi}\, \sin \frac{n\pi x}{2}\, dx \right\}$$
$$= \frac{1}{2}\, \frac{4}{n^2\pi^2} \left[\cos \frac{n\pi x}{2} \right]_{-1}^{2} = \frac{2}{n^2\pi^2} \left[\cos(n\pi) - \cos \frac{n\pi}{2} \right]$$
$$= \frac{2}{n^2\pi^2} \left[(-1)^n - \cos \frac{n\pi}{2} \right], \quad n = 1, 2, \ldots,$$

$$b_n = \frac{1}{2} \int_{-2}^{2} f(x) \sin \frac{n\pi x}{2}\, dx = \frac{1}{2} \int_{-1}^{2} (1+x) \sin \frac{n\pi x}{2}\, dx$$
$$= \frac{1}{2} \left\{ \left[(1+x) \left(-\frac{2}{n\pi} \right) \cos \frac{n\pi x}{2} \right]_{-1}^{2} + \int_{-1}^{2} \frac{2}{n\pi}\, \cos \frac{n\pi x}{2}\, dx \right\}$$
$$= \frac{1}{2} \left\{ -\frac{6}{n\pi}\, \cos(n\pi) + \frac{4}{n^2\pi^2} \left[\sin \frac{n\pi x}{2} \right]_{-1}^{2} \right\}$$
$$= (-1)^{n+1}\, \frac{3}{n\pi} + \frac{4}{n^2\pi^2}\, \sin \frac{n\pi}{2}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{9}{8} + \sum_{n=1}^{n} \left\{ \frac{2}{n^2\pi^2} \left[(-1)^n - \cos\frac{n\pi}{2} \right] \cos\frac{n\pi x}{2} \right.$$
$$\left. + \left[(-1)^{n+1} \frac{3}{n\pi} + \frac{4}{n^2\pi^2} \sin\frac{n\pi}{2} \right] \sin\frac{n\pi x}{2} \right\},$$

$$\text{series} = \begin{cases} 0, & -2 < x \leq -1, \\ 1 + x, & -1 < x < 2, \\ \frac{3}{2}, & x = -2,\, 2. \end{cases}$$