3. (i) As in Example 8.10,

$$\begin{aligned}
&\mathcal{F}_S[u](\omega,t) = U(\omega,t),\\
&\mathcal{F}_S[u_{xx}](\omega,t) = \frac{2}{\pi}\,\omega u(0,t) - \omega^2\mathcal{F}_S[u](\omega,t) = \frac{2\omega}{\pi} - \omega^2 U(\omega,t),\\
&\mathcal{F}_S[e^{-x}] = \frac{2\omega}{\pi(1+\omega^2)}\\
\Rightarrow\quad & U'(\omega,t) + 4\omega^2 U(\omega,t) = \frac{8\omega}{\pi}, \quad U(\omega,0) = \frac{2\omega}{\pi(1+\omega^2)}\\
\Rightarrow\quad & U(\omega,t) = Ce^{-4\omega^2 t} + \frac{2}{\pi\omega} = \left[\frac{2\omega}{\pi(1+\omega^2)} - \frac{2}{\pi\omega}\right]e^{-4\omega^2 t} + \frac{2}{\pi\omega}\\
\Rightarrow\quad & u(x,t) = \frac{2}{\pi}\int_0^\infty \left[\left(\frac{\omega}{1+\omega^2} - \frac{1}{\omega}\right)e^{-4\omega^2 t} + \frac{1}{\omega}\right]\sin(\omega x)\,d\omega.
\end{aligned}$$

An alternative form of the solution can be obtained if the equation for U is solved by means of an integrating factor:

$$\begin{aligned}
U(\omega,t) &= e^{-4\omega^2 t}\left(\int_0^t e^{4\omega^2\tau}\,\frac{8\pi}{\pi}\,d\tau + C\right)\\
&= \frac{8}{\pi}\int_0^t \omega e^{-\omega^2\cdot 4(t-\tau)}\,d\tau + \frac{2\omega}{\pi(1+\omega^2)}\,e^{-4\omega^2 t}\\
\Rightarrow\quad u(x,t) &= \frac{8}{\pi}\int_0^t \mathcal{F}_S^{-1}\big[\omega e^{-\omega^2\cdot 4(t-\tau)}\big]\,d\tau + \frac{2}{\pi}\,\mathcal{F}_S^{-1}\left[\frac{\omega}{1+\omega^2}\,e^{-4\omega^2 t}\right]\\
&= \frac{8}{\pi}\int_0^t \frac{2\sqrt{\pi}}{64(t-\tau)^{3/2}}\,xe^{-x^2/[16(t-\tau)]}\,d\tau + \frac{2}{\pi}\,\mathcal{F}_S^{-1}\left[\frac{\omega}{1+\omega^2}\,e^{-4\omega^2 t}\right]\\
&= \frac{x}{4\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{3/2}}\,e^{-x^2/[16(t-\tau)]}\,d\tau + \frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\,e^{-4\omega^2 t}\sin(\omega x)\,d\omega.
\end{aligned}$$

(ii) As in (i),

$$\mathcal{F}_S[u](\omega,t) = U(\omega,t),$$

$$\mathcal{F}_S[u_{xx}](\omega,t) = \frac{2}{\pi}\,\omega u(0,t) - \omega^2 \mathcal{F}_S[u](\omega,t) = \frac{2\omega}{\pi}\,e^{-3t} - \omega^2 U(\omega,t),$$

$$\mathcal{F}_S\left[\,\frac{2x}{x^2+9}\,\right] = 2e^{-3\omega}$$

$$\Rightarrow\quad U'(\omega,t) + \omega^2 U(\omega,t) = \frac{2\omega}{\pi}\,e^{-3t},\quad U(\omega,0) = 2e^{-3\omega}$$

$$\Rightarrow\quad U(\omega,t) = Ce^{-\omega^2 t} + \frac{2\omega}{\pi(\omega^2-3)}\,e^{-3t}$$

$$= \left[2e^{-3\omega} - \frac{2\omega}{\pi(\omega^2-3)}\,\right]e^{-\omega t} + \frac{2\omega}{\pi(\omega^2-3)}\,e^{-3t}$$

$$\Rightarrow\quad u(x,t) = \mathcal{F}_S^{-1}[U](x,t)$$

$$= 2\int_0^\infty \left\{\left[e^{-3\omega} - \frac{\omega}{\pi(\omega^2-3)}\,\right]e^{-\omega t} + \frac{\omega}{\pi(\omega^2-3)}\,e^{-3t}\right\}\sin(\omega x)\,d\omega.$$

Alternatively,

$$U(\omega,t) = e^{-\omega^2 t}\left(\int_0^t e^{\omega^2\tau}\,\frac{2\omega}{\pi}\,e^{-3\tau}\,d\tau + C\right)$$

$$= \frac{2}{\pi}\int_0^t \omega e^{-\omega^2(t-\tau)}\,e^{-3\tau}\,d\tau + 2e^{-3\omega}e^{-\omega^2 t}$$

$$\Rightarrow\quad u(x,t) = \frac{2}{\pi}\int_0^t \mathcal{F}_S^{-1}\big[\omega e^{-\omega^2(t-\tau)}\big]e^{-3\tau}\,d\tau + 2\mathcal{F}_S^{-1}\big[e^{-3\omega}\,e^{-\omega^2 t}\big]$$

$$= \frac{2}{\pi}\int_0^t \frac{2\sqrt{\pi}}{8(t-\tau)^{3/2}}\,xe^{-x^2/[4(t-\tau)]}\,e^{-3\tau}\,d\tau + 2\mathcal{F}_S^{-1}\big[e^{-\omega^2 t-3\omega}\big]$$

$$= \frac{x}{2\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{3/2}}\,e^{-x^2/[4(t-\tau)]}e^{-3\tau}\,d\tau + 2\int_0^\infty e^{-\omega^2 t-3\omega}\sin(\omega x)\,d\omega.$$

4. (i) As in Example 8.11,

$$\mathcal{F}_C[u](\omega,t) = U(\omega,t),$$

$$\mathcal{F}_C[u_{xx}](\omega,t) = -\frac{2}{\pi}\,u_x(0,t) - \omega^2\mathcal{F}_C[u](\omega,t) = \frac{4}{\pi} - \omega^2 U(\omega,t),$$

$$\mathcal{F}_C[2e^{-x}] = 2\,\frac{2}{\pi}\,\frac{1}{(1+\omega^2)} = \frac{4}{\pi(1+\omega^2)}$$

$$\Rightarrow \quad U'(\omega,t) + 4\omega^2 U(\omega,t) = \frac{16}{\pi}, \quad U(\omega,0) = \frac{4}{\pi(1+\omega^2)}$$

$$\Rightarrow \quad U(\omega,t) = Ce^{-4\omega^2 t} + \frac{4}{\pi\omega^2} = \left[\frac{4}{\pi(1+\omega^2)} - \frac{4}{\pi\omega^2}\right]e^{-4\omega^2 t} + \frac{4}{\pi\omega^2}$$

$$\Rightarrow \quad u(x,t) = \mathcal{F}_C^{-1}[U](x,t) = \frac{4}{\pi}\int_0^\infty \left[\left(\frac{1}{1+\omega^2} - \frac{1}{\omega^2}\right)e^{-4\omega^2 t} + \frac{1}{\omega^2}\right]\cos(\omega x)\,d\omega.$$

Alternatively,

$$U(\omega,t) = e^{-4\omega^2 t}\left(\int_0^t e^{4\omega^2\tau}\,\frac{16}{\pi}\,d\tau + C\right)$$

$$= \frac{16}{\pi}\int_0^t e^{-\omega^2\cdot 4(t-\tau)}\,d\tau + \frac{4}{\pi(1+\omega^2)}\,e^{-4\omega^2 t}$$

$$\Rightarrow \quad u(x,t) = \frac{16}{\pi}\int_0^t \mathcal{F}_C^{-1}\left[e^{-\omega^2\cdot 4(t-\tau)}\right]d\tau + \frac{4}{\pi}\,\mathcal{F}_C^{-1}\left[\frac{1}{1+\omega^2}\,e^{-4\omega^2 t}\right]$$

$$= \frac{16}{\pi}\int_0^t \frac{\sqrt{\pi}}{4(t-\tau)^{1/2}}\,e^{-x^2/[16(t-\tau)]}\,d\tau + \frac{4}{\pi}\,\mathcal{F}_C^{-1}\left[\frac{1}{1+\omega^2}\,e^{-4\omega^2 t}\right]$$

$$= \frac{4}{\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}}\,e^{-x^2/[16(t-\tau)]}\,d\tau + \frac{4}{\pi}\int_0^\infty \frac{1}{1+\omega^2}\,e^{-4\omega^2 t}\cos(\omega x)\,d\omega.$$

(ii) As in (i),

$$\mathcal{F}_C[u](\omega,t) = U(\omega,t),$$

$$\mathcal{F}_C[u_{xx}](\omega,t) = -\frac{2}{\pi}\,u_x(0,t) - \omega^2\mathcal{F}_C[u](\omega,t) = -\frac{6}{\pi}\,e^{-2t} - \omega^2 U(\omega,t),$$

$$\mathcal{F}_C\left[\frac{4}{x^2+9}\right] = \tfrac{4}{3}\,e^{-3\omega}$$

$$\Rightarrow \quad U'(\omega,t) + \omega^2 U(\omega,t) = -\frac{6}{\pi}\,e^{-2t}, \quad U(\omega,0) = \tfrac{4}{3}\,e^{-3\omega}$$

$$\Rightarrow \quad U(\omega,t) = Ce^{-\omega^2 t} - \frac{6}{\pi(\omega^2-2)}\,e^{-2t}$$

$$= \left[\tfrac{4}{3}\,e^{-3\omega} + \frac{6}{\pi(\omega^2-2)}\right]e^{-\omega^2 t} - \frac{6}{\pi(\omega^2-2)}\,e^{-2t}$$

$$\Rightarrow \quad u(x,t) = \mathcal{F}_C^{-1}[U](x,t)$$

$$= 2\int_0^\infty \left\{\left[\tfrac{2}{3}\,e^{-3\omega} + \frac{3}{\pi(\omega^2-2)}\right]e^{-\omega^2 t} - \frac{3}{\pi(\omega^2-2)}\,e^{-2t}\right\}\cos(\omega x)\,d\omega.$$

Alternatively,

$$U(\omega,t) = e^{-\omega^2 t}\left[\int_0^t e^{\omega^2\tau}\left(-\frac{6}{\pi}\right)e^{-2\tau}\,d\tau + C\right]$$

$$= -\frac{6}{\pi}\int_0^t e^{-\omega^2(t-\tau)}\,e^{-2\tau}\,d\tau + \tfrac{4}{3}\,e^{-3\omega}\,e^{-\omega^2 t}$$

$$\Rightarrow \quad u(x,t) = -\frac{6}{\pi}\int_0^t e^{-2\tau}\mathcal{F}_C^{-1}\left[e^{-\omega^2(t-\tau)}\right]d\tau + \tfrac{4}{3}\,\mathcal{F}_C^{-1}\left[e^{-\omega^2 t-3\omega}\right]$$

$$= -\frac{6}{\pi}\int_0^t \frac{\sqrt{\pi}}{2(t-\tau)^{1/2}}\,e^{-x^2/[4(t-\tau)]}\,e^{-2\tau}\,d\tau + \tfrac{4}{3}\,\mathcal{F}_C^{-1}\left[e^{-\omega^2 t-3\omega}\right]$$

$$= -\frac{3}{\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}}\,e^{-x^2/[4(t-\tau)]}\,e^{-2\tau}\,d\tau + \frac{4}{3}\int_0^\infty e^{-\omega^2 t-3\omega}\cos(\omega x)\,d\omega.$$