Chapter 8

1. (i) By definition,

$$F(\omega) = \mathcal{F}[f](\omega) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = \frac{1}{2\pi}\int\limits_{-1}^{1} xe^{i\omega x}\,dx$$

$$= \frac{1}{2\pi}\left\{\left[x\,\frac{1}{i\omega}\,e^{i\omega x}\right]_{-1}^{1} - \int\limits_{-1}^{1}\frac{1}{i\omega}\,e^{i\omega x}\,dx\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1}{i\omega}\,e^{i\omega} + \frac{1}{i\omega}\,e^{-i\omega} - \frac{1}{i^2\omega^2}\left[e^{i\omega x}\right]_{-1}^{1}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1}{i\omega}\,2\cos\omega + \frac{1}{\omega^2}\left(e^{i\omega} - e^{-i\omega}\right)\right\}$$

$$= \frac{1}{2\pi}\left\{-\frac{i}{\omega}\,2\cos\omega + \frac{1}{\omega^2}\,2i\sin\omega\right\} = \frac{i}{\pi}\left\{\frac{\sin\omega}{\omega^2} - \frac{\cos\omega}{\omega}\right\}.$$

(ii) Similarly,

$$F(\omega) = \mathcal{F}[f](\omega) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = \frac{1}{2\pi}\int\limits_{-1}^{1} e^{x}e^{i\omega x}\,dx$$

$$= \frac{1}{2\pi}\int\limits_{-1}^{1} e^{(1+i\omega)x}\,dx = \frac{1}{2\pi}\,\frac{1}{1+i\omega}\left[e^{(1+i\omega)x}\right]_{-1}^{1}$$

$$= \frac{1-i\omega}{2\pi(1+\omega^2)}\left[e(\cos\omega + i\sin\omega) - e^{-1}(\cos\omega - i\sin\omega)\right]$$

$$= \frac{1-i\omega}{2\pi(1+\omega^2)}\left[(e-e^{-1})\cos\omega + i(e+e^{-1})\sin\omega\right]$$

$$= \frac{1-i\omega}{2\pi(1+\omega^2)}\left(2\sinh 1\cos\omega + 2i\cosh 1\sin\omega\right)$$

$$= \frac{1}{\pi(1+\omega^2)}\left(\sinh 1\cos\omega + \omega\cosh 1\sin\omega\right)$$

$$+ \frac{i}{\pi(1+\omega^2)}\left(\cosh 1\sin\omega - \omega\sinh 1\cos\omega\right).$$

2. (i) Using Table A2, we have

$$\begin{aligned}
&\mathcal{F}[u](\omega,t) = U(\omega,t), \quad \mathcal{F}[u_{xx}](\omega,t) = -\omega^2 U(\omega,t),\\
&\mathcal{F}[3e^{-2x^2}] = \frac{3}{2\sqrt{2\pi}}\, e^{-\omega^2/8}\\
\Rightarrow\quad & U''(\omega,t) + 4\omega^2 U(\omega,t) = 0, \quad U(\omega,0) = \frac{3}{2\sqrt{2\pi}}\, e^{-\omega^2/8}, \quad U'(\omega,0) = 0\\
\Rightarrow\quad & U(\omega,t) = C_1(\omega)\cos(2\omega t) + C_2(\omega)\sin(2\omega t) = \frac{3}{2\sqrt{2\pi}}\, e^{-\omega^2/8}\cos(2\omega t)\\
&= \frac{3}{2\sqrt{2\pi}}\,\frac{1}{2}\,(e^{2i\omega t} + e^{-2i\omega t})e^{-\omega^2/8}\\
\Rightarrow\quad & u(x,t) = \mathcal{F}^{-1}[U](x,t) = \int_{-\infty}^{\infty} \frac{3}{4\sqrt{2\pi}}\,(e^{2i\omega t} + e^{-2i\omega t})e^{-\omega^2/8}e^{-i\omega x}\,d\omega\\
&= \frac{3}{2}\int_{-\infty}^{\infty} \frac{1}{2\sqrt{2\pi}}\, e^{-\omega^2/8}\big[e^{-i\omega(x-2t)} + e^{-i\omega(x+2t)}\big]\,d\omega\\
&= \tfrac{3}{2}\left[e^{-2(x-2t)} + e^{-2(x+2t)}\right].
\end{aligned}$$

(ii) As in (i),

$$\begin{aligned}
&\mathcal{F}[u](\omega,t) = U(\omega,t), \quad \mathcal{F}[u_{xx}](\omega,t) = -\omega^2 U(\omega,t),\\
&\mathcal{F}\left[\frac{1}{x^2+9}\right] = \tfrac{1}{6}\, e^{-3|\omega|}\\
\Rightarrow\quad & U''(\omega,t) + \tfrac{1}{9}\,\omega^2 U(\omega,t) = 0, \quad U(\omega,0) = \tfrac{1}{6}\, e^{-3|\omega|}, \quad U'(\omega,t) = 0\\
\Rightarrow\quad & U(\omega,t) = \tfrac{1}{6}\, e^{-3|\omega|}\cos\left(\tfrac{1}{3}\,\omega t\right) = \tfrac{1}{6}\, e^{-3|\omega|}\,\tfrac{1}{2}(e^{i\omega t/3} + e^{-i\omega t/3})\\
\Rightarrow\quad & u(x,t) = \mathcal{F}^{-1}[U](x,t) = \frac{1}{2}\int_{-\infty}^{\infty} \tfrac{1}{6}\, e^{-3|\omega|}\big(e^{i\omega t/3} + e^{-i\omega t/3}\big)e^{-i\omega x}\,d\omega\\
&= \frac{1}{2}\int_{-\infty}^{\infty} \tfrac{1}{6}\, e^{-3|\omega|}\big[e^{-i\omega(x-t/3)} + e^{-i\omega(x+t/3)}\big]\,d\omega\\
&= \frac{1}{2}\left[\frac{1}{(x-t/3)^2+9} + \frac{1}{(x+t/3)^2+9}\right].
\end{aligned}$$