6. (i) As in Example 7.7,

$$L = 1, \quad K = 2, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad q(x,y) = \sum_{n=1}^{\infty} q_n(y)\sin(n\pi x) \quad \Rightarrow \quad q_2 = 1, \quad q_n = 0, \ n \neq 2,$$

$$f_1(x) = \sum_{n=1}^{\infty} f_{1n}\sin(n\pi x) \quad \Rightarrow \quad f_{11} = 1, \quad f_{13} = -2, \quad f_{1n} = 0, \ n \neq 1, 3,$$

$$f_2(x) = \sum_{n=1}^{\infty} f_{2n}\sin(n\pi x) \quad \Rightarrow \quad f_{22} = -1, \quad f_{2n} = 0, \ n \neq 2$$

$$\Rightarrow \quad c_1''(y) - \pi^2 c_1(y) = 0, \quad c_1(0) = 1, \quad c_1(2) = 0$$

$$\Rightarrow \quad c_1(y) = C_1\sinh(\pi y) + C_2\sinh\big(\pi(y-2)\big) = -\operatorname{cosech}(2\pi)\sinh\big(\pi(y-2)\big),$$

$$c_2''(y) - 4\pi^2 c_2(y) = 1, \quad c_2(0) = 0, \quad c_2(2) = -1$$

$$\Rightarrow \quad c_2(y) = C_1\sinh(2\pi y) + C_2\sinh\big(2\pi(y-2)\big) - \frac{1}{4\pi^2}$$

$$= \left(\frac{1}{4\pi^2} - 1\right)\operatorname{cosech}(4\pi)\sin(2\pi y)$$

$$- \frac{1}{4\pi^2}\operatorname{cosech}(4\pi)\sin\big(2\pi(y-2)\big) - \frac{1}{4\pi^2},$$

$$c_3''(y) - 9\pi^2 c_3(y) = 0, \quad c_3(0) = -2, \quad c_3(2) = 0$$

$$\Rightarrow \quad c_3(y) = C_1\sinh(3\pi y) + C_2\sinh\big(3\pi(y-2)\big) = 2\operatorname{cosech}(6\pi)\sinh\big(3\pi(y-2)\big),$$

$$c_n''(y) - n^2\pi^2 c_n(y) = 0, \quad c_n(0) = 0, \quad c_n(2) = 0 \quad \Rightarrow \quad c_n(y) \equiv 0, \quad n \neq 1, 2, 3$$

$$\begin{aligned}
\Rightarrow \quad u(x,y) = {} & -\operatorname{cosech}(2\pi)\sinh\big(\pi(y-2)\big)\sin(\pi x) \\
& + \Bigg[\left(\frac{1}{4\pi^2} - 1\right)\operatorname{cosech}(4\pi)\sinh(2\pi y) \\
& \qquad - \frac{1}{4\pi^2}\operatorname{cosech}(4\pi)\sinh\big(2\pi(y-2)\big) - \frac{1}{4\pi^2}\Bigg]\sin(2\pi x) \\
& + 2\operatorname{cosech}(6\pi)\sinh\big(3\pi(y-2)\big)\sin(3\pi x).
\end{aligned}$$

(ii) Using (7.8), here we have

$$L = 1, \quad K = 2, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad q(x,y) = \sum_{n=1}^{\infty} q_n(y)\sin(n\pi x),$$

$$q_n(y) = \frac{2}{L}\int_0^L q(x,y)\sin(n\pi x)\,dx = 2\int_0^1 \tfrac{1}{2}\,\pi^3\sin(n\pi x)\,dx$$

$$= \pi^3\left(-\frac{1}{n\pi}\right)\big[\cos(n\pi x)\big]_0^1 = \big[1-(-1)^n\big]\,\frac{\pi^2}{n}, \quad n = 1,2,\ldots,$$

$$f_1(x) = \sum_{n=1}^{\infty} f_{1n}\sin(n\pi x), \quad f_{1n} = \frac{2}{L}\int_0^1 f_1(x)\sin(n\pi x)\,dx = 0, \quad n = 1,2,\ldots,$$

$$f_2(x) = \sum_{n=1}^{\infty} f_{2n}\sin(n\pi x),$$

$$f_{2n} = \frac{2}{L}\int_0^L f_2(x)\sin(n\pi x)\,dx = 2\int_0^1 \tfrac{1}{2}\,x\sin(n\pi x)\,dx$$

$$= \left[x\left(-\frac{1}{n\pi}\right)\cos(n\pi x)\right]_0^1 + \int_0^1 \frac{1}{n\pi}\,\cos(n\pi x)\,dx$$

$$= -\frac{1}{n\pi}\,\cos(n\pi) + \frac{1}{n^2\pi^2}\,\big[\sin(n\pi x)\big]_0^1 = (-1)^{n+1}\frac{1}{n\pi}, \quad n = 1,2,\ldots$$

$$\Rightarrow \quad c_n''(y) - n^2\pi^2 c_n(y) = \big[1-(-1)^n\big]\,\frac{\pi^2}{n}, \quad c_n(0) = 0, \quad c_n(2) = (-1)^{n+1}\frac{1}{n\pi}$$

$$\Rightarrow \quad c_n(y) = C_1\sinh(n\pi y) + C_2\sinh\big(n\pi(y-2)\big) - \big[1-(-1)^n\big]\,\frac{1}{n^3}$$

$$= \left\{\big[1-(-1)^n\big]\,\frac{1}{n^3} + (-1)^{n+1}\,\frac{1}{n\pi}\right\}\operatorname{cosech}(2n\pi)\sinh(n\pi y)$$

$$+ \big[(-1)^n - 1\big]\,\frac{1}{n^3}\,\operatorname{cosech}(2n\pi)\sinh\big(n\pi(y-2)\big) - \big[1-(-1)^n\big]\,\frac{1}{n^3}$$

$$\Rightarrow \quad u(x,y) = \sum_{n=1}^{\infty}\left\{\left[\frac{1-(-1)^n}{n^3} + \frac{(-1)^{n+1}}{n\pi}\right]\operatorname{cosech}(2n\pi)\sinh(n\pi y)\right.$$

$$+ \frac{(-1)^n - 1}{n^3}\,\operatorname{cosech}(2n\pi)\sinh\big(n\pi(y-2)\big)$$

$$\left. - \frac{1-(-1)^n}{n^3}\right\}\sin(n\pi x).$$

(iii) We have

$$L=1,\quad K=2,\quad \lambda_n=\frac{n^2\pi^2}{2},\quad Y_n(y)=\sin\frac{n\pi y}{2},\quad n=1,2,\ldots$$

$$\Rightarrow\quad q(x,y)=\sum_{n=1}^{\infty} q_n(x)\sin\frac{n\pi y}{2}\quad\Rightarrow\quad q_2=-\pi^2,\quad q_n=0,\ n\neq 2,$$

$$g_1(y)=\sum_{n=1}^{\infty} g_{1n}\sin\frac{n\pi y}{2}\quad\Rightarrow\quad g_{12}=2,\quad g_{1n}=0,\ n\neq 2,$$

$$g_2(y)=\sum_{n=2}^{\infty} g_{2n}\sin\frac{n\pi y}{2}\quad\Rightarrow\quad g_{21}=1,\quad g_{2n}=0,\ n\neq 1$$

$$\Rightarrow\quad c_1''(x)-\frac{\pi^2}{4}\,c_1(x)=0,\quad c_1(0)=0,\quad c_1(1)=1$$

$$\Rightarrow\quad c_1(x)=C_1\sinh\frac{\pi x}{2}+C_2\sinh\frac{\pi(x-1)}{2}=\operatorname{cosech}\frac{\pi}{2}\,\sinh\frac{\pi x}{2},$$

$$c_2''(x)-\pi^2 c_2(x)=-\pi^2,\quad c_2(0)=2,\quad c_2(1)=0$$

$$\Rightarrow\quad c_2(x)=C_1\sinh(\pi x)+C_2\sinh\big(\pi(x-1)\big)+1$$

$$=-\operatorname{cosech}\pi\big[\sinh(\pi x)+\sinh\big(\pi(x-1)\big)\big]+1,$$

$$c_n''(x)+\frac{n^2\pi^2}{4}\,c_n(x)=0,\quad c_n(0)=0,\quad c_n(1)=0\quad\Rightarrow\quad c_n(x)\equiv 0,\quad n\neq 1,2$$

$$\Rightarrow\quad u(x,y)=\operatorname{cosech}\frac{\pi}{2}\,\sinh\frac{\pi x}{2}\,\sin\frac{\pi y}{2}$$

$$+\big\{1-(\operatorname{cosech}\pi)\big[\sinh(\pi x)+\sinh\big(\pi(x-1)\big)\big]\big\}\sin(\pi y).$$

7. (i) We have (see Example 7.8)

$$\lambda_0=0,\quad \Theta_0(\theta)\equiv 1,$$

$$\lambda_n=n^2,\quad \Theta_{1n}(\theta)=\cos(n\theta),\quad \Theta_{2n}(\theta)=\sin(n\theta),\quad n=1,2,\ldots$$

$$\Rightarrow\quad q(r)=q_0(r)+\sum_{n=1}^{\infty}\big[q_{1n}(r)\cos(n\theta)+q_{2n}(r)\sin(n\theta)\big]$$

$$\Rightarrow\quad q_0=-8,\quad q_{1n}=q_{2n}=0,\ n=1,2,\ldots,$$

$$f(\theta)=f_0+\sum_{n=1}^{\infty}\big[f_{1n}\cos(n\theta)+f_{2n}\sin(n\theta)\big]$$

$$\Rightarrow\quad f_0=-1,\quad f_{13}=2,\quad f_{1n}=0,\ n\neq 3,\quad f_{21}=2,\quad f_{2n}=0,\ n\neq 1$$

$$\Rightarrow \quad c_0''(r) + r^{-1}c_0'(r) = -8, \quad c_0(1) = -1, \quad c_0(0) \text{ bounded}$$
$$\Rightarrow \quad c_0(r) = -2r^2 + C_1 \ln r + C_2 = -2r^2 + 1,$$
$$c_{13}''(r) + r^{-1}c_{13}'(r) - 9r^{-2}c_{13}(r) = 0, \quad c_{13}(1) = 2, \quad c_{13}(0) \text{ bounded}$$
$$\Rightarrow \quad c_{13}(r) = C_1 r^3 + C_2 r^{-3} = 2r^3,$$
$$c_{1n}''(r) + r^{-1}c_{1n}'(r) - r^{-2}c_{1n}(r) = 0, \quad c_{1n}(1) = 0, \quad c_{1n}(0) \text{ bounded}$$
$$\Rightarrow \quad c_{1n}(r) \equiv 0, \ n \neq 3,$$
$$c_{21}''(r) + r^{-1}c_{21}'(r) - r^{-2}c_{21}(r) = 0, \quad c_{21}(1) = 2, \quad c_{21}(0) \text{ bounded}$$
$$\Rightarrow \quad c_{21}(r) = C_1 r + C_2 r^{-1} = 2r,$$
$$c_{2n}''(r) + r^{-1}c_{2n}'(r) - r^{-2}c_{2n}(r) = 0, \quad c_{2n}(1) = 0, \quad c_{2n}(0) \text{ bounded}$$
$$\Rightarrow \quad c_{2n}(r) \equiv 0, \ n \neq 1$$
$$\Rightarrow \quad u(r,\theta) = 1 - 2r^2 + 2r\sin\theta + 2r^3\cos(3\theta).$$

(ii) As in (i),

$$\lambda_0 = 0, \quad \Theta_0(\theta) \equiv 1,$$
$$\lambda_n = n^2, \quad \Theta_{1n}(\theta) = \cos(n\theta), \quad \Theta_{2n}(\theta) = \sin(n\theta), \quad n = 1, 2, \ldots$$
$$\Rightarrow \quad q(r) = q_0(r) + \sum_{n=1}^{\infty} \left[q_{1n}(r)\cos(n\theta) + q_{2n}(r)\sin(n\theta) \right]$$
$$\Rightarrow \quad q_0 = 9r, \quad q_{11} = 6, \quad q_{1n} = 0, \ n \neq 1, \quad q_{21} = 3, \quad q_{2n} = 0, \ n \neq 1,$$
$$f(\theta) = f_0 + \sum_{n=1}^{\infty} \left[f_{1n}\cos(n\theta) + f_{2n}\sin(n\theta) \right]$$
$$\Rightarrow \quad f_0 = 1, \quad f_{1n} = 0, \ n = 1, 2, \ldots, \quad f_{21} = -3, \quad f_{2n} = 0, \ n \neq 1$$
$$\Rightarrow \quad c_0''(r) + r^{-1}c_0'(r) = 9r, \quad c_0(1) = 1, \quad c_0(0) \text{ bounded}$$
$$\Rightarrow \quad c_0(r) = r^3 + C_1 \ln r + C_2 = r^3,$$
$$c_{11}''(r) + r^{-1}c_{11}'(r) - r^{-2}c_{11}(r) = 6, \quad c_{11}(1) = 0, \quad c_{11}(0) \text{ bounded}$$
$$\Rightarrow \quad c_{11}(r) = C_1 r + C_2 r^{-1} + 2r^2 = -2r + 2r^2,$$
$$c_{1n}''(r) + r^{-1}c_{1n}'(r) - r^{-2}c_{1n}(r) = 0, \quad c_{1n}(1) = 0, \quad c_{1n}(0) \text{ bounded}$$
$$\Rightarrow \quad c_{1n}(r) \equiv 0, \ n \neq 1,$$
$$c_{21}''(r) + r^{-1}c_{21}'(r) - r^{-2}c_{21}(r) = 3, \quad c_{21}(1) = -3, \quad c_{21}(0) \text{ bounded}$$

$$
\begin{aligned}
\Rightarrow\quad & c_{21}(r) = C_1 r + C_2 r^{-1} + r^2 = -4r + r^2, \\
& c_{2n}''(r) + r^{-1} c_{2n}'(r) - r^{-2} c_{2n}(r) = 0, \quad c_{2n}(1) = 0, \quad c_{2n}(0) \text{ bounded} \\
\Rightarrow\quad & c_{2n}(r) \equiv 0,\ n \neq 1 \\
\Rightarrow\quad & u(r,\theta) = r^3 + (2r^2 - 2r)\cos\theta + (r^2 - 4r)\sin\theta.
\end{aligned}
$$