4. (i) Here

$$L = 1, \quad c = 1, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x) \quad \Rightarrow \quad q_2 = 2, \quad q_n = 0, \ n \neq 2,$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x) \quad \Rightarrow \quad f_1 = 1, \quad f_n = 0, \ n \neq 1,$$

$$g(x) = \sum_{n=1}^{\infty} g_n \sin(n\pi x) \quad \Rightarrow \quad g_2 = -3, \quad g_n = 0, \ n \neq 2$$

$$\Rightarrow \quad c_1''(t) + \pi^2 c_1(t) = 0, \quad c_1(0) = 1, \quad c_1'(0) = 0$$

$$\Rightarrow \quad c_1(t) = C_1\cos(\pi t) + C_2\sin(\pi t) = \cos(\pi t),$$

$$c_2''(t) + 4\pi^2 c_2(t) = 2, \quad c_2(0) = 0, \quad c_2'(0) = -3$$

$$\Rightarrow \quad c_2(t) = C_1\cos(2\pi t) + C_2\sin(2\pi t) + \frac{1}{2\pi^2} = -\frac{1}{2\pi^2}\cos(2\pi t) - \frac{3}{2\pi}\sin(2\pi t) + \frac{1}{2\pi^2},$$

$$c_n''(t) + n^2\pi^2 c_n(t) = 0, \quad c_n(0) = 0, \quad c_n'(0) = 0 \quad \Rightarrow \quad c_n(t) \equiv 0, \quad n \neq 1, 2$$

$$\Rightarrow \quad u(x,t) = \cos(\pi t)\sin(\pi x) + \left[\frac{1}{2\pi^2} - \frac{1}{2\pi^2}\cos(2\pi t) - \frac{3}{2\pi}\sin(2\pi t)\right]\sin(2\pi x).$$

(ii) As in (i),

$$L = 1, \quad c = 1, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x) \quad \Rightarrow \quad q_1 = t, \quad q_n = 0, \ n \neq 1,$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x) \quad \Rightarrow \quad f_1 = 1, \quad f_n = 0, \ n \neq 1,$$

$$g(x) = \sum_{n=1}^{\infty} g_n \sin(n\pi x) \quad \Rightarrow \quad g_1 = 2, \quad g_3 = 4, \quad g_n = 0, \ n \neq 1, 3$$

$$\Rightarrow \quad c_1''(t) + \pi^2 c_1(t) = t, \quad c_1(0) = 1, \quad c_1'(0) = 2$$

$$\Rightarrow \quad c_1(t) = C_1\cos(\pi t) + C_2\sin(\pi t) + \frac{1}{\pi^2}t = \cos(\pi t) + \left(\frac{2}{\pi} - \frac{1}{\pi^3}\right)\sin(\pi t) + \frac{1}{\pi^2}t,$$

$$c_3''(t) + 9\pi^2 c_3(t) = 0, \quad c_3(0) = 0, \quad c_3'(0) = 4$$

$$\Rightarrow \quad c_3(t) = C_1\cos(3\pi t) + C_2\sin(3\pi t) = \frac{4}{3\pi}\,\sin(3\pi t),$$

$$c_n''(t) + n^2\pi^2 c_n(t) = 0, \quad c_n(0) = 0, \quad c_n'(0) = 0 \quad \Rightarrow \quad c_n(t) \equiv 0, \quad n \neq 1,3$$

$$\Rightarrow \quad u(x,t) = \left[\cos(\pi t) + \left(\frac{2}{\pi} - \frac{1}{\pi^3}\right)\sin(\pi t) + \frac{1}{\pi^2}\,t\right]\sin(\pi x) + \frac{4}{3\pi}\,\sin(3\pi t)\sin(3\pi x).$$

(iii) Here

$$L = 1, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad q_n(t) = \frac{2}{L}\int_0^L qX_n\,dx = 2\int_0^1 \tfrac{1}{2}\,(x-1)t\sin(n\pi x)\,dx$$

$$= t\left\{\left[(x-1)\left(-\frac{1}{n\pi}\right)\cos(n\pi x)\right]_0^1 + \int_0^1 \frac{1}{n\pi}\,\cos(n\pi x)\,dx\right\}$$

$$= t\left\{-\frac{1}{n\pi} + \frac{1}{n^2\pi^2}\,\big[\sin(n\pi x)\big]_0^1\right\} = -\frac{1}{n\pi}\,t, \quad n = 1, 2, \ldots,$$

$$f_n = \frac{2}{L}\int_0^L fX_n\,dx = 2\int_0^1 \sin(n\pi x)\,dx$$

$$= 2\left(-\frac{1}{n\pi}\right)\big[\cos(n\pi x)\big]_0^1 = \big[1 - (-1)^n\big]\,\frac{2}{n\pi}, \quad g_n = 0, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad c_n''(t) + n^2\pi^2 c_n(t) = -\frac{1}{n\pi}\,t, \quad c_n(0) = \big[1 - (-1)^n\big]\,\frac{2}{n\pi}, \quad c_n'(0) = 0$$

$$\Rightarrow \quad c_n(t) = C_1\cos(n\pi t) + C_2\sin(n\pi t) - \frac{1}{n^3\pi^3}\,t$$

$$\Rightarrow \quad c_n'(t) = -C_1 n\pi\sin(n\pi t) + C_2 n\pi\cos(n\pi t) - \frac{1}{n^3\pi^3}.$$

The ICs now lead to

$$\big[1 - (-1)^n\big]\,\frac{2}{n\pi} = C_1, \quad 0 = C_2 n\pi - \frac{1}{n^3\pi^3}$$

$$\Rightarrow \quad c_n(t) = \big[1 - (-1)^n\big]\,\frac{2}{n\pi}\,\cos(n\pi t) + \frac{1}{n^4\pi^4}\,\sin(n\pi t) - \frac{1}{n^3\pi^3}\,t$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty}\left\{\big[1 - (-1)^n\big]\,\frac{2}{n\pi}\,\cos(n\pi t) + \frac{1}{n^4\pi^4}\,\sin(n\pi t) - \frac{1}{n^3\pi^3}\,t\right\}\sin(n\pi x).$$

5. (i) Here

$$L=1,\quad c=1,\quad \lambda_n=n^2\pi^2,\quad X_n(x)=\cos(n\pi x),\quad n=0,1,2,\dots$$

$$\Rightarrow\quad q(x,t)=\sum_{n=0}^{\infty} q_n(t)\cos(n\pi x)\quad\Rightarrow\quad q_0=3,\quad q_n=0,\ n\neq 0,$$

$$f(x)=\sum_{n=0}^{\infty} f_n\cos(n\pi x)\quad\Rightarrow\quad f_0=1,\quad f_2=2,\quad f_n=0,\ n\neq 0,2,$$

$$g(x)=\sum_{n=0}^{\infty} g_n\cos(n\pi x)\quad\Rightarrow\quad g_3=1,\quad g_n=0,\ n\neq 3$$

$$\Rightarrow\quad c_0''(t)=3,\quad c_0(0)=1,\quad c_0'(0)=0\quad\Rightarrow\quad c_0(t)=\tfrac{3}{2}\,t^2+1,$$

$$c_2''(t)+4\pi^2c_2(t)=0,\quad c_2(0)=2,\quad c_2'(0)=0$$

$$\Rightarrow\quad c_2(t)=C_1\cos(2\pi t)+C_2\sin(2\pi t)=2\cos(2\pi t),$$

$$c_3''(t)+9\pi^2c_3(t)=0,\quad c_3(0)=0,\quad c_3'(0)=1$$

$$\Rightarrow\quad c_3(t)=C_1\cos(3\pi t)+C_2\sin(3\pi t)=\frac{1}{3\pi}\,\sin(3\pi t),$$

$$c_n''(t)+n^2\pi^2c_n(t)=0,\quad c_n(0)=0,\quad c_n'(0)=0\quad\Rightarrow\quad c_n(t)\equiv 0,\quad n\neq 0,2,3$$

$$\Rightarrow\quad u(x,t)=\tfrac{3}{2}\,t^2+1+2\cos(2\pi t)\cos(2\pi x)+\frac{1}{3\pi}\,\sin(3\pi t)\cos(3\pi x).$$

(ii) As in (i),

$$L=1,\quad c=1,\quad \lambda_n=n^2\pi^2,\quad X_n(x)=\cos(n\pi x),\quad n=0,1,2,\dots$$

$$\Rightarrow\quad q(x,t)=\sum_{n=0}^{\infty} q_n(t)\cos(n\pi x)\quad\Rightarrow\quad q_0=1,\quad q_2=t,\quad q_n=0,\ n\neq 0,2,$$

$$f(x)=\sum_{n=0}^{\infty} f_n\cos(n\pi x)\quad\Rightarrow\quad f_2=2,\quad f_n=0,\ n\neq 2,$$

$$g(x)=\sum_{n=0}^{\infty} g_n\cos(n\pi x)\quad\Rightarrow\quad g_0=1,\quad g_1=1,\quad g_2=-1,\quad g_n=0,\ n\neq 0,1,2$$

$$\Rightarrow\quad c_0''(t)=1,\quad c_0(0)=0,\quad c_0'(0)=1\quad\Rightarrow\quad c_0(t)=\tfrac{1}{2}\,t^2+t,$$

$$c_1''(t)+\pi^2c_1(t)=0,\quad c_1(0)=0,\quad c_1'(0)=1$$

$$\Rightarrow\quad c_1(t)=C_1\cos(\pi t)+C_2\sin(\pi t)=\frac{1}{\pi}\,\sin(\pi t),$$

$$c_2''(t)+4\pi^2c_2(t)=t,\quad c_2(0)=2,\quad c_2'(0)=-1$$

$$
\begin{aligned}
\Rightarrow \quad c_2(t) &= C_1 \cos(2\pi t) + C_2 \sin(2\pi t) + \frac{1}{4\pi^2}\, t \\
&= 2\cos(2\pi t) - \left(\frac{1}{2\pi} + \frac{1}{8\pi^3}\right)\sin(2\pi t) + \frac{1}{4\pi^2}\, t, \\
c_n''(t) + n^2\pi^2 c_n(t) = 0, &\quad c_n(0) = 0, \quad c_n'(0) = 0 \quad \Rightarrow \quad c_n(t) \equiv 0, \quad n \neq 0, 1, 2 \\
\Rightarrow \quad u(x,t) &= \tfrac{1}{2}\, t^2 + t + \frac{1}{\pi}\sin(\pi t)\cos(\pi x) \\
&\qquad + \left[2\cos(2\pi t) - \left(\frac{1}{2\pi} + \frac{1}{8\pi^3}\right)\sin(2\pi t) + \frac{1}{4\pi^2}\, t\right]\cos(2\pi x).
\end{aligned}
$$