Chapter 7

1. (i) As in Example 7.1,

$$L = 1, \quad k = 1, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x) \quad \Rightarrow \quad q_2 = 2t, \quad q_n = 0,\ n \neq 2,$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x) \quad \Rightarrow \quad f_2 = 1, \quad f_4 = -5, \quad f_n = 0,\ n \neq 2, 4$$

$$\Rightarrow \quad c_2'(t) + 4\pi^2 c_2(t) = 2t, \quad c_2(0) = 1$$

$$\Rightarrow \quad c_2(t) = Ce^{-4\pi^2 t} + \frac{1}{2\pi^2}\,t - \frac{1}{8\pi^4} = \left(1 + \frac{1}{8\pi^4}\right)e^{-4\pi^2 t} + \frac{1}{2\pi^2}\,t - \frac{1}{8\pi^4},$$

$$c_4'(t) + 16\pi^2 c_4(t) = 0, \quad c_4(0) = -5 \quad \Rightarrow \quad c_4(t) = Ce^{-16\pi^2 t} = -5e^{-16\pi^2 t},$$

$$c_n'(t) + n^2\pi^2 c_n(t) = 0, \quad c_n(0) = 0 \quad \Rightarrow \quad c_n(t) \equiv 0, \quad n \neq 2, 4$$

$$\Rightarrow \quad u(x,t) = \left[\left(1 + \frac{1}{8\pi^4}\right)e^{-4\pi^2 t} + \frac{1}{2\pi^2}\,t - \frac{1}{8\pi^4}\right]\sin(2\pi x) - 5e^{-16\pi^2 t}\sin(4\pi x).$$

(ii) Here again

$$L = 1, \quad k = 1, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x) \quad \Rightarrow \quad q_3 = e^{-t}, \quad q_5 = -1, \quad q_n = 0,\ n \neq 3, 5,$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x) \quad \Rightarrow \quad f_1 = 1, \quad f_3 = 2, \quad f_n = 0,\ n \neq 1, 3$$

$$\Rightarrow \quad c_1'(t) + \pi^2 c_1(t) = 0, \quad c_1(0) = 1 \quad \Rightarrow \quad c_1(t) = Ce^{-\pi^2 t} = e^{-\pi^2 t},$$

$$c_3'(t) + 9\pi^2 c_3(t) = e^{-t}, \quad c_3(0) = 2$$

$$\Rightarrow \quad c_3(t) = Ce^{-9\pi^2 t} + \frac{1}{9\pi^2 - 1}\,e^{-t} = \frac{18\pi^2 - 3}{9\pi^2 - 1}\,e^{-9\pi^2 t} + \frac{1}{9\pi^2 - 1}\,e^{-t},$$

$$c_5'(t) + 25\pi^2 c_5(t) = -1, \quad c_5(0) = 0$$

$$\Rightarrow \quad c_5(t) = Ce^{-25\pi^2 t} - \frac{1}{25\pi^2} = \frac{1}{25\pi^2}\left(e^{-25\pi^2 t} - 1\right),$$

$$c_n'(t)+n^2\pi^2c_n(t)=0,\quad c_n(0)=0\quad\Rightarrow\quad c_n(t)\equiv 0,\quad n\neq 1,3,5$$

$$\Rightarrow\quad u(x,t)=e^{-\pi^2t}\sin(\pi x)+\left(\frac{18\pi^2-3}{9\pi^2-1}\,e^{-9\pi^2t}+\frac{1}{9\pi^2-1}\,e^{-t}\right)\sin(3\pi x)$$
$$+\frac{1}{25\pi^2}\left(e^{-25\pi^2t}-1\right)\sin(5\pi x).$$

(iii) We have

$$L=1,\quad k=1,\quad \lambda_n=n^2\pi^2,\quad X_n(x)=\sin(n\pi x),\quad n=1,2,\dots$$
$$\Rightarrow\quad q(x,t)=\sum_{n=1}^{\infty}q_n(t)\sin(n\pi x)\quad\Rightarrow\quad q_1=t-1,\quad q_n=0,\ n\neq 1,$$
$$f(x)=\sum_{n=1}^{\infty}f_n\sin(n\pi x)\quad\Rightarrow\quad f_1=1,\quad f_2=2,\quad f_n=0,\ n\neq 1,2$$
$$\Rightarrow\quad c_1'(t)+\pi^2c_1(t)=t-1,\quad c_1(0)=1$$
$$\Rightarrow\quad c_1(t)=Ce^{-\pi^2t}+\frac{1}{\pi^2}\,t-\frac{1}{\pi^2}-\frac{1}{\pi^4}=\left(1+\frac{1}{\pi^2}+\frac{1}{\pi^4}\right)e^{-\pi^2t}+\frac{1}{\pi^2}\,t-\frac{1}{\pi^2}-\frac{1}{\pi^4},$$
$$c_2'(t)+4\pi^2c_2(t)=0,\quad c_2(0)=2\quad\Rightarrow\quad c_2(t)=Ce^{-4\pi^2t}=2e^{-4\pi^2t},$$
$$c_n'(t)+n^2\pi^2c_n(t)=0,\quad c_n(0)=0\quad\Rightarrow\quad c_n(t)\equiv 0,\quad n\neq 1,2$$
$$\Rightarrow\quad u(x,t)=\left[\left(1+\frac{1}{\pi^2}+\frac{1}{\pi^4}\right)e^{-\pi^2t}+\frac{1}{\pi^2}\,t-\frac{1}{\pi^2}-\frac{1}{\pi^4}\right]\sin(\pi x)+2e^{-4\pi^2t}\sin(2\pi x).$$

(iv) Once more,

$$L=1,\quad k=1,\quad \lambda_n=n^2\pi^2,\quad X_n(x)=\sin(n\pi x),\quad n=1,2,\dots$$
$$\Rightarrow\quad q(x,t)=\sum_{n=1}^{\infty}q_n(t)\sin(n\pi x),\quad f(x)=\sum_{n=1}^{\infty}f_n\sin(n\pi x),$$
$$\int_0^1 q(x,t)\sin(n\pi x)\,dx=\tfrac{1}{2}\,t\int_0^1(x-1)\sin(n\pi x)\,dx$$
$$=\tfrac{1}{2}\,t\left\{\left[(x-1)\left(-\frac{1}{n\pi}\right)\cos(n\pi x)\right]_0^1+\int_0^1\frac{1}{n\pi}\cos(n\pi x)\,dx\right\}$$
$$=\tfrac{1}{2}\,t\left\{-\frac{1}{n\pi}+\frac{1}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^1\right\}=-\frac{1}{2n\pi}\,t,$$

$$\int_0^1 \sin^2(n\pi x)\,dx = \tfrac{1}{2} \quad \Rightarrow \quad q_n(t) = -\frac{1}{n\pi}\,t$$

$$\begin{aligned}
\int_0^1 f(x)\sin(n\pi x)\,dx &= \int_0^1 x\sin(n\pi x)\,dx \\
&= \left[x\Big(-\frac{1}{n\pi}\Big)\cos(n\pi x)\right]_0^1 + \int_0^1 \frac{1}{n\pi}\,\cos(n\pi x)\,dx \\
&= -\frac{1}{n\pi}\,\cos(n\pi) + \frac{1}{n^2\pi^2}\,\big[\sin(n\pi x)\big]_0^1 = (-1)^{n+1}\frac{1}{n\pi}
\end{aligned}$$

$$\Rightarrow \quad f_n = (-1)^{n+1}\frac{2}{n\pi}$$

$$\Rightarrow \quad c_n'(t) + n^2\pi^2 c_n(t) = -\frac{1}{n\pi}\,t, \quad c_n(0) = (-1)^{n+1}\frac{2}{n\pi}, \quad n = 1, 2, \ldots$$

$$\begin{aligned}
\Rightarrow \quad c_n(t) &= Ce^{-n^2\pi^2 t} - \frac{1}{n^3\pi^3}\,t + \frac{1}{n^5\pi^5} \\
&= \left[(-1)^{n+1}\frac{2}{n\pi} - \frac{1}{n^5\pi^5}\right]e^{-n^2\pi^2 t} - \frac{1}{n^3\pi^3}\,t + \frac{1}{n^5\pi^5}
\end{aligned}$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty}\left\{\left[(-1)^{n+1}\frac{2}{n\pi} - \frac{1}{n^5\pi^5}\right]e^{-n^2\pi^2 t} - \frac{1}{n^3\pi^3}\,t + \frac{1}{n^5\pi^5}\right\}\sin(n\pi x).$$

2. (i) As in Example 7.3,

$$L = 1, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \cos(n\pi x), \quad n = 0, 1, 2, \ldots$$

$$\Rightarrow \quad q(x,t) = \sum_{n=0}^{\infty} q_n(t)\cos(n\pi x) \quad \Rightarrow \quad q_0 = 2, \quad q_2 = 1, \quad q_n = 0, \; n \neq 0, 2,$$

$$f(x) = \sum_{n=0}^{\infty} f_n \cos(n\pi x) \quad \Rightarrow \quad f_1 = 2, \quad f_2 = -1, \quad f_n = 0, \; n \neq 1, 2$$

$$\Rightarrow \quad c_0'(t) = 2, \quad c_0(0) = 0 \quad \Rightarrow \quad c_0(t) = 2t,$$

$$c_1'(t) + \pi^2 c_1(t) = 0, \quad c_1(0) = 2 \quad \Rightarrow \quad c_1(t) = Ce^{-\pi^2 t} = 2e^{-\pi^2 t},$$

$$c_2'(t) + 4\pi^2 c_2(t) = 1, \quad c_2(0) = -1$$

$$\Rightarrow \quad c_2(t) = Ce^{-4\pi^2 t} + \frac{1}{4\pi^2} = -\Big(1 + \frac{1}{4\pi^2}\Big)e^{-4\pi^2 t} + \frac{1}{4\pi^2},$$

$$c_n'(t) + n^2\pi^2 c_n(t) = 0, \quad c_n(0) = 0 \quad \Rightarrow \quad c_n(t) \equiv 0, \quad n \neq 0, 1, 2$$

$$\Rightarrow \quad u(x,t) = 2t + 2e^{-\pi^2 t}\cos(\pi x) + \left[\frac{1}{4\pi^2} - \left(1 + \frac{1}{4\pi^2}\right)e^{-4\pi^2 t}\right]\cos(2\pi x).$$

(ii) We have

$$L = 1, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \cos(n\pi x), \quad n = 0, 1, 2, \dots$$

$$\Rightarrow \quad q(x,t) = \sum_{n=0}^{\infty} q_n(t)\cos(n\pi x) \quad \Rightarrow \quad q_0 = t, \quad q_1 = -t, \quad q_n = 0, \ n \neq 0, 1,$$

$$f(x) = \sum_{n=0}^{\infty} f_n \cos(n\pi x) \quad \Rightarrow \quad f_0 = 1, \quad f_4 = 3, \quad f_n = 0, \ n \neq 0, 4$$

$$\Rightarrow \quad c_0'(t) = t, \quad c_0(0) = 1 \quad \Rightarrow \quad c_0(t) = \tfrac{1}{2}t^2 + 1,$$

$$c_1'(t) + \pi^2 c_1(t) = -t, \quad c_1(0) = 0$$

$$\Rightarrow \quad c_1(t) = Ce^{-\pi^2 t} - \frac{1}{\pi^2}t + \frac{1}{\pi^4} = -\frac{1}{\pi^4}e^{-\pi^2 t} - \frac{1}{\pi^2}t + \frac{1}{\pi^4},$$

$$c_4'(t) + 16\pi^2 c_4(t) = 0, \quad c_4(0) = 3 \quad \Rightarrow \quad c_4(t) = Ce^{-16\pi^2 t} = 3e^{-16\pi^2 t},$$

$$c_n'(t) + n^2\pi^2 c_n(t) = 0, \quad c_n(0) = 0 \quad \Rightarrow \quad c_n(0) \equiv 0, \quad n \neq 0, 1, 4$$

$$\Rightarrow \quad u(x,t) = \tfrac{1}{2}t^2 + 1 + \left(\frac{1}{\pi^4} - \frac{1}{\pi^2}t - \frac{1}{\pi^4}e^{-\pi^2 t}\right)\cos(\pi x) + 3e^{-16\pi^2 t}\cos(4\pi x).$$

(iii) Here

$$L = 1, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \cos(n\pi x), \quad n = 0, 1, 2, \dots$$

$$\Rightarrow \quad q(x,t) = \sum_{n=0}^{\infty} q_n(t)\cos(n\pi x) \quad \Rightarrow \quad q_1 = e^{-t}, \quad q_n = 0, \ n \neq 1,$$

$$f(x) = \sum_{n=0}^{\infty} f_n \cos(n\pi x) \quad \Rightarrow \quad f_0 = 2, \quad f_3 = -1, \quad f_n = 0, \ n \neq 0, 3$$

$$\Rightarrow \quad c_0'(t) = 0, \quad c_0(0) = 2 \quad \Rightarrow \quad c_0(t) \equiv 2,$$

$$c_1'(t) + \pi^2 c_1(t) = e^{-t}, \quad c_1(0) = 0$$

$$\Rightarrow \quad c_1(t) = Ce^{-\pi^2 t} + \frac{1}{\pi^2 - 1}e^{-t} = \frac{1}{\pi^2 - 1}(e^{-t} - e^{-\pi^2 t}),$$

$$c_3'(t) + 9\pi^2 c_3(t) = 0, \quad c_3(0) = -1 \quad \Rightarrow \quad c_3(t) = Ce^{-9\pi^2 t} = -e^{-9\pi^2 t},$$

$$c_n'(t) + n^2\pi^2 c_n(t) = 0, \quad c_n(0) = 0 \quad \Rightarrow \quad c_n(t) \equiv 0, \quad n \neq 0, 1, 3$$

$$\Rightarrow \quad u(x,t) = 2 + \frac{1}{\pi^2 - 1}(e^{-t} - e^{-\pi^2 t})\cos(\pi x) - e^{-9\pi^2 t}\cos(3\pi x).$$

(iv) As in (iii),

$$L = 1, \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \cos(n\pi x), \quad n = 0, 1, 2, \dots$$

$$\Rightarrow \quad q(x,t) = \sum_{n=0}^{\infty} q_n(t)\cos(n\pi x), \quad f(x) = \sum_{n=0}^{\infty} f_n \cos(n\pi x),$$

$$\int_0^1 q(x,t)\,dx = \int_0^1 (1-x)t\,dx = t\big[x - \tfrac{1}{2}\,x^2\big]_0^1 = \tfrac{1}{2}\,t,$$

$$\int_0^1 q(x,t)\cos(n\pi x)\,dx = \int_0^1 (1-x)t\cos(n\pi x)\,dx$$

$$= t\Bigg\{\Bigg[(1-x)\,\frac{1}{n\pi}\,\sin(n\pi x)\Bigg]_0^1 - \int_0^1 \frac{1}{n\pi}\,\sin(n\pi x)(-dx)\Bigg\}$$

$$= t\Bigg(-\frac{1}{n^2\pi^2}\Bigg)\big[\cos(n\pi x)\big]_0^1 = \big[1-(-1)^n\big]\,\frac{1}{n^2\pi^2}\,t, \quad n = 1, 2, \dots,$$

$$\int_0^1 \cos^2(n\pi x)\,dx = \tfrac{1}{2}$$

$$\Rightarrow \quad q_0(t) = \tfrac{1}{2}\,t, \quad q_n(t) = \big[1-(-1)^n\big]\,\frac{2}{n^2\pi^2}\,t, \quad n = 1, 2, \dots,$$

$$\int_0^1 f(x)\,dx = \int_0^1 x\,dx = \tfrac{1}{2},$$

$$\int_0^1 f(x)\cos(n\pi x)\,dx = \int_0^1 x\cos(n\pi x)\,dx = \Bigg[x\,\frac{1}{n\pi}\,\sin(n\pi x)\Bigg]_0^1 - \int_0^1 \frac{1}{n\pi}\,\sin(n\pi x)\,dx$$

$$= \frac{1}{n^2\pi^2}\,\big[\cos(n\pi x)\big]_0^1 = \big[(-1)^n - 1\big]\,\frac{1}{n^2\pi^2}, \quad n = 1, 2, \dots$$

$$\Rightarrow \quad f_0 = \tfrac{1}{2}, \quad f_n = \big[(-1)^n - 1\big]\,\frac{2}{n^2\pi^2}, \quad n = 1, 2, \dots$$

$$\Rightarrow \quad c_0'(t) = \tfrac{1}{2}\,t, \quad c_0(0) = \tfrac{1}{2} \quad \Rightarrow \quad c_0(t) = \tfrac{1}{4}\,t^2 + \tfrac{1}{2},$$

$$c_n'(t) + n^2\pi^2 c_n(t) = \big[1-(-1)^n\big]\,\frac{2}{n^2\pi^2}\,t, \quad c_n(0) = \big[(-1)^n - 1\big]\,\frac{2}{n^2\pi^2}$$

$$\Rightarrow \quad c_n(t) = \big[1-(-1)^n\big]\,\frac{2}{n^2\pi^2}\left[\left(\frac{1}{n^4\pi^4}-1\right)e^{-n^2\pi^2 t}+\frac{1}{n^2\pi^2}\,t-\frac{1}{n^4\pi^4}\right], \quad n=1,2,\ldots$$

$$\Rightarrow \quad u(x,t) = \tfrac{1}{4}\,t^2+\tfrac{1}{2}+\sum_{n=1}^{\infty}\big[1-(-1)^n\big]\,\frac{2}{n^2\pi^2}\left[\left(\frac{1}{n^4\pi^4}-1\right)e^{-n^2\pi^2 t}\right.$$

$$\left.+\frac{1}{n^2\pi^2}\,t-\frac{1}{n^4\pi^4}\right]\cos(n\pi x).$$

3. (i) Here

$$L=1, \quad \lambda_n=\frac{(2n-1)^2\pi^2}{4}, \quad X_n(x)=\sin\frac{(2n-1)\pi x}{2}, \quad n=1,2,\ldots$$

$$\Rightarrow \quad q(x,t)=\sum_{n=1}^{\infty}q_n(t)\sin\frac{(2n-1)\pi x}{2} \quad \Rightarrow \quad q_2=1, \quad q_3=-2, \quad q_n=0,\ n\neq 2,3,$$

$$f(x)=\sum_{n=1}^{\infty}f_n\sin\frac{(2n-1)\pi x}{2} \quad \Rightarrow \quad f_2=1, \quad f_n=0,\ n\neq 2$$

$$\Rightarrow \quad c_2'(t)+\frac{9\pi^2}{4}\,c_2(t)=1, \quad c_2(0)=1$$

$$\Rightarrow \quad c_2(t)=Ce^{-9\pi^2 t/4}+\frac{4}{9\pi^2}=\left(1-\frac{4}{9\pi^2}\right)e^{-9\pi^2 t/4}+\frac{4}{9\pi^2},$$

$$c_3'(t)+\frac{25\pi^2}{4}\,c_3(t)=-2, \quad c_3(0)=0$$

$$\Rightarrow \quad c_3(t)=Ce^{-25\pi^2 t/4}-\frac{8}{25\pi^2}=\frac{8}{25\pi^2}\,e^{-25\pi^2 t/4}-\frac{8}{25\pi^2},$$

$$c_n'(t)+\frac{(2n-1)^2\pi^2}{4}\,c_n(t)=0, \quad c_n(0)=0 \quad \Rightarrow \quad c_n(t)\equiv 0, \quad n\neq 2,3$$

$$\Rightarrow \quad u(x,t)=\left[\left(1-\frac{4}{9\pi^2}\right)e^{-9\pi^2 t/4}+\frac{4}{9\pi^2}\right]\sin\frac{3\pi x}{2}$$

$$+\frac{8}{25\pi^2}\big(e^{-25\pi^2 t/4}-1\big)\sin\frac{5\pi x}{2}.$$

(ii) As in (i),

$$L=1, \quad \lambda_n=\frac{(2n-1)^2\pi^2}{4}, \quad X_n(x)=\sin\frac{(2n-1)\pi x}{2}, \quad n=1,2,\ldots$$

$$\Rightarrow \quad q(x,t)=\sum_{n=1}^{\infty}q_n(t)\sin\frac{(2n-1)\pi x}{2} \quad \Rightarrow \quad q_1=t, \quad q_n=0,\ n\neq 1,$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\frac{(2n-1)\pi x}{2} \quad \Rightarrow \quad f_1 = 1, \quad f_3 = 2, \quad f_n = 0, \; n \neq 1, 3$$

$$\Rightarrow \quad c_1'(t) + \frac{\pi^2}{4}\, c_1(t) = t, \quad c_1(0) = 1$$

$$\Rightarrow \quad c_1(t) = Ce^{-\pi^2 t/4} + \frac{4}{\pi^2}\, t - \frac{16}{\pi^4} = \left(1 + \frac{16}{\pi^4}\right) e^{-\pi^2 t/4} + \frac{4}{\pi^2}\, t - \frac{16}{\pi^4},$$

$$c_3'(t) + \frac{25\pi^2}{4}\, c_3(t) = 0, \quad c_3(0) = 2$$

$$\Rightarrow \quad c_3(t) = Ce^{-25\pi^2 t/4} = 2e^{-25\pi^2 t/4},$$

$$c_n'(t) + \frac{(2n-1)^2\pi^2}{4}\, c_n(t) = 0, \quad c_n(0) = 0 \quad \Rightarrow \quad c_n(t) \equiv 0, \quad n \neq 1, 3$$

$$\Rightarrow \quad u(x,t) = \left[\left(1 + \frac{16}{\pi^4}\right) e^{-\pi^2 t/4} + \frac{4}{\pi^2}\, t - \frac{16}{\pi^4}\right] \sin\frac{\pi x}{2} + 2e^{-25\pi^2 t/4} \sin\frac{5\pi x}{2}.$$