8. (i) We are using the solution form $u(x,y) = X(x)Y(y)$ in the PDE and the three homogeneous BCs, and follow the procedure described in Section 5.3 with $L = 1$ and $K = 2$:

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad \Rightarrow \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda = \text{const},$$

$$X(x)Y(0) = 0, \quad 0 < x < 1,$$

$$X(0)Y(y) = 0, \quad X(1)Y(y) = 0, \quad 0 < y < 2$$

$$\Rightarrow \quad X'' + \lambda X = 0, \quad X(0) = X(1) = 0, \qquad Y'' - \lambda Y = 0, \quad Y(0) = 0$$

$$\Rightarrow \quad \lambda_n = n^2\pi^2, \quad X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots,$$

$$Y_n(y) = C_1 \cosh(n\pi y) + C_2 \sinh(n\pi y) = C_2 \sinh(n\pi y), \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \sinh(n\pi y).$$

From the nonhomogeneous BC we now obtain

$$u(x,2) = x = \sum_{n=1}^{\infty} c_n \sinh(2n\pi) \sin(n\pi x),$$

$$c_n \sinh(2n\pi) = 2 \int_0^1 x \sin(n\pi x)\, dx$$

$$= 2\left\{ \left[x\left(-\frac{1}{n\pi}\right) \cos(n\pi x) \right]_0^1 + \int_0^1 \frac{1}{n\pi} \cos(n\pi x)\, dx \right\}$$

$$= -\frac{2}{n\pi} \cos(n\pi) + \frac{2}{n^2\pi^2} \left[\sin(n\pi x) \right]_0^1$$

$$= (-1)^{n+1} \frac{2}{n\pi}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,y) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi \sinh(2n\pi)} \sin(n\pi x) \sinh(n\pi y).$$

(ii) By analogy with (i), here we have

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad \Rightarrow \quad -\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = -\lambda = \text{const}$$

$$
\begin{aligned}
& X(x)Y(0) = 0, \quad X(x)Y(2) = 0, \quad 0 < x < 1, \\
& X(1)Y(y) = 0, \quad 0 < y < 2 \\
\Rightarrow \quad & X'' - \lambda X = 0, \quad X(1) = 0, \qquad Y'' + \lambda Y = 0, \quad Y(0) = Y(2) = 0 \\
\Rightarrow \quad & Y(y) = C_1 \cos(\sqrt{\lambda}\, y) + C_2 \sin(\sqrt{\lambda}\, y) = C_2 \sin(\sqrt{\lambda}\, y), \quad \lambda > 0 \\
\Rightarrow \quad & \lambda_n = \frac{n^2\pi^2}{4}, \quad Y_n(y) = \sin \frac{n\pi y}{2}, \quad n = 1, 2, \ldots, \\
& X_n(x) = C_1 \cosh \frac{n\pi(x-1)}{2} + C_2 \sinh \frac{n\pi(x-1)}{2} \\
& \qquad = C_2 \sinh \frac{n\pi(x-1)}{2}, \quad n = 1, 2, \ldots \\
\Rightarrow \quad & u(x,y) = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi(x-1)}{2} \, \sin \frac{n\pi y}{2} \\
\Rightarrow \quad & u(0,y) = y = \sum_{n=1}^{\infty} -c_n \sinh \frac{n\pi}{2} \, \sin \frac{n\pi y}{2}, \\
& - c_n \sinh \frac{n\pi}{2} = \int_0^2 y \sin \frac{n\pi y}{2} \, dy = \left\{ \left[y \left(- \frac{2}{n\pi} \right) \cos \frac{n\pi y}{2} \right]_0^2 + \int_0^2 \frac{2}{n\pi} \, \cos \frac{n\pi y}{2} \, dy \right\} \\
& \qquad = -\frac{4}{n\pi} \, \cos(n\pi) + \frac{4}{n^2\pi^2} \left[\sin \frac{n\pi y}{2} \right]_0^2 = (-1)^{n+1} \frac{4}{n\pi}, \quad n = 1, 2, \ldots \\
\Rightarrow \quad & u(x,y) = \sum_{n=1}^{\infty} (-1)^n \frac{4}{n\pi \sinh(n\pi/2)} \, \sinh \frac{n\pi(x-1)}{2} \, \sin \frac{n\pi y}{2}.
\end{aligned}
$$

(iii) As in (ii),

$$
\begin{aligned}
& X''(x)Y(y) + X(x)Y''(y) = 0 \quad \Rightarrow \quad -\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = -\lambda = \text{const} \\
& X(x)Y(0) = 0, \quad X(x)Y(2) = 0, \quad 0 < x < 1, \\
& X(0)Y(y) = 0, \quad 0 < y < 2 \\
\Rightarrow \quad & X'' - \lambda X = 0, \quad X(0) = 0, \qquad Y'' + \lambda Y = 0, \quad Y(0) = Y(2) = 0 \\
\Rightarrow \quad & \lambda_n = \frac{n^2\pi^2}{4}, \quad Y_n(y) = \sin \frac{n\pi y}{2}, \quad n = 1, 2, \ldots, \\
& X_n(x) = C_1 \cosh \frac{n\pi x}{2} + C_2 \sinh \frac{n\pi x}{2} = C_2 \sinh \frac{n\pi x}{2}, \quad n = 1, 2, \ldots
\end{aligned}
$$

$$\Rightarrow \quad u(x,y) = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi x}{2} \, \sin \frac{n\pi y}{2}$$

$$\Rightarrow \quad u(1,y) = y = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi}{2} \, \sin \frac{n\pi y}{2},$$

$$c_n \sinh \frac{n\pi}{2} = \int_0^2 y \sin \frac{n\pi y}{2} \, dy = (-1)^{n+1} \frac{4}{n\pi}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad u(x,y) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n\pi \sinh(n\pi/2)} \, \sinh \frac{n\pi x}{2} \, \sin \frac{n\pi y}{2}.$$