Chapter 1

1. (i) This is a variables separable equation and $y(x) \equiv 0$ is a solution. If $y(x) \not\equiv 0$, then

$$\frac{dy}{y} = \frac{2x}{x^2+1}\,dx \quad \Rightarrow \quad \ln|y| = \ln(x^2+1) + C_0 \quad \Rightarrow \quad y(x) = C(x^2+1).$$

The above formula is the general solution, since it also includes the solution $y(x) \equiv 0$.

(ii) This is a linear equation, solved by the integrating factor method:

$$\begin{aligned}
&\frac{dy}{dx} + \frac{2}{x-1}\,y = \frac{x}{x-1} \\
\Rightarrow\quad &\mu = \exp\left\{\int \frac{2}{x-1}\,dx\right\} = e^{2\ln|x-1|} = (x-1)^2 \\
\Rightarrow\quad &y(x) = \frac{1}{(x-1)^2}\int (x-1)^2\frac{x}{x-1}\,dx = \frac{1}{(x-1)^2}\int (x^2-x)\,dx \\
&\qquad = (x-1)^{-2}\big(\tfrac{1}{3}\,x^3 - \tfrac{1}{2}\,x^2 + C\big).
\end{aligned}$$

(iii) This is a homogeneous first-order linear equation with constant coefficients, solved by means of the corresponding characteristic equation:

$$2s + 5 = 0 \quad \Rightarrow \quad s = -\tfrac{5}{2} \quad \Rightarrow \quad y(x) = C\,e^{-5x/2}.$$

(iv) This is a homogeneous second-order linear equation with constant coefficients, so

$$s^2 - 4s + 3 = 0 \quad \Rightarrow \quad s_1 = 1,\ \ s_2 = 3 \quad \Rightarrow \quad y(x) = C_1e^x + C_2e^{3x}.$$

(v) As in (iv),

$$4s^2 + 4s + 1 = 0 \quad \Rightarrow \quad s_1 = s_2 = -\tfrac{1}{2} \quad \Rightarrow \quad y(x) = (C_1 + C_2x)e^{-x/2}.$$

(vi) We have

$$s^2 + 2s + 5 = 0 \quad \Rightarrow \quad s_{1,2} = -1 \pm 2i \quad \Rightarrow \quad y(x) = e^{-x}\big[C_1\cos(2x) + C_2\sin(2x)\big].$$

(vii) This is a nonhomogeneous linear equation with constant coefficients, so we use the "complementary function + particular integral" method:

$$\begin{aligned}
& s+2=0 \quad \Rightarrow \quad s=-2 \quad \Rightarrow \quad y_{CF}=Ce^{-2x}, \\
& y_{PI}=ax+b+ce^{4x} \\
\Rightarrow \quad & (a+4ce^{4x})+2(ax+b+ce^{4x})=2x+e^{4x} \\
\Rightarrow \quad & 2ax+(a+2b)+6ce^{4x}=2x+e^{4x} \\
\Rightarrow \quad & 2a=2, \quad a+2b=0, \quad 6c=1 \\
\Rightarrow \quad & a=1, \quad b=-\tfrac{1}{2}, \quad c=\tfrac{1}{6} \quad \Rightarrow \quad y_{PI}=x-\tfrac{1}{2}+\tfrac{1}{6}\,e^{4x} \\
\Rightarrow \quad & y(x)=Ce^{-2x}+x-\tfrac{1}{2}+\tfrac{1}{6}\,e^{4x}.
\end{aligned}$$

(viii) Here

$$\begin{aligned}
& 2s-1=0 \quad \Rightarrow \quad s=\tfrac{1}{2} \quad \Rightarrow \quad y_{CF}=Ce^{x/2}, \\
& y_{PI}=axe^{x/2} \quad \Rightarrow \quad 2a\big(e^{x/2}+\tfrac{1}{2}\,xe^{x/2}\big)-axe^{x/2}=2ae^{x/2}=e^{x/2} \\
\Rightarrow \quad & 2a=1 \quad \Rightarrow \quad a=\tfrac{1}{2} \quad \Rightarrow \quad y_{PI}=\tfrac{1}{2}\,xe^{x/2} \\
\Rightarrow \quad & y(x)=Ce^{x/2}+\tfrac{1}{2}\,xe^{x/2}=\big(C+\tfrac{1}{2}\,x\big)e^{x/2}.
\end{aligned}$$

(ix) As in (vii),

$$\begin{aligned}
& s^2-1=0 \quad \Rightarrow \quad s_{1,2}=\pm 1 \quad \Rightarrow \quad y_{CF}=C_1\cosh x+C_2\sinh x, \\
& y_{PI}=ax^2+bx+c \\
\Rightarrow \quad & 2a-(ax^2+bx+c)=-ax^2-bx+(2a-c)=x^2-x+2 \\
\Rightarrow \quad & a=-1, \quad b=1, \quad c=-4 \quad \Rightarrow \quad y_{PI}=-x^2+x-4 \\
\Rightarrow \quad & y(x)=C_1\cosh x+C_2\sinh x-x^2+x-4.
\end{aligned}$$

(x) As in (viii),

$$\begin{aligned}
& s^2-25=0 \quad \Rightarrow \quad s_{1,2}=\pm 5 \quad \Rightarrow \quad y_{CF}=C_1e^{5x}+C_2e^{-5x}, \\
& y_{PI}=axe^{-5x} \quad \Rightarrow \quad a(-10e^{-5x}+25xe^{-5x})-25axe^{-5x}=-10ae^{-5x}=30e^{-5x} \\
\Rightarrow \quad & -10a=30 \quad \Rightarrow \quad a=-3 \quad \Rightarrow \quad y_{PI}=-3xe^{-5x} \\
& y(x)=C_1e^{5x}+C_2e^{-5x}-3xe^{-5x}=C_1e^{5x}+(C_2-3x)e^{-5x}.
\end{aligned}$$

2. (i) For any functions y_1, y_2 and any numbers c_1, c_2

$$\begin{aligned} & x(c_1y_1 + c_2y_2)'' - (c_1y_1 + c_2y_2)' \sin x \\ & \qquad = c_1(xy_1'' - y_1' \sin x) + c_2(xy_2'' - y_2' \sin x) \\ \Rightarrow \quad & \text{equation is linear.} \end{aligned}$$

(ii) For any functions y_1, y_2 and any numbers c_1, c_2, in general

$$\begin{aligned} & (c_1y_1 + c_2y_2)' + 2x\sin(c_1y_1 + c_2y_2) = c_1y_1' + c_2y_2' + 2x\sin(c_1y_1 + c_2y_2) \\ & \qquad \neq c_1(y_1' + 2x\sin y_1) + c_2(y_2' + 2x\sin y_2) \\ & \qquad\quad = c_1y_1' + 2c_1x\sin y_1 + c_2y_2' + 2c_2x\sin y_2 \\ \Rightarrow \quad & \text{equation is nonlinear.} \end{aligned}$$

(iii) As in (ii),

$$\begin{aligned} & (c_1y_1 + c_2y_2)'(c_1y_1 + c_2y_2)'' - x(c_1y_1 + c_2y_2) \\ & \qquad = c_1^2y_1'y_1'' + c_1c_2(y_1'y_2'' + y_1''y_2') + c_2^2y_2'y_2'' - c_1(xy_1) - c_2(xy_2) \\ & \qquad\quad \neq c_1(y_1'y_1'' - xy_1) + c_2(y_2'y_2'' - xy_2) \\ \Rightarrow \quad & \text{equation is nonlinear.} \end{aligned}$$

(iv) As is (i),

$$\begin{aligned} & (c_1y_1 + c_2y_2)'' + \sqrt{x}\,(c_1y_1 + c_2y_2) = c_1(y_1'' + \sqrt{x}\,y_1) + c_2(y_2'' + \sqrt{x}\,y_2) \\ \Rightarrow \quad & \text{equation is linear.} \end{aligned}$$