FORMULAS AND TABLES

Orthogonality Formulas

For all m, $n = 1, 2, \ldots,$

$$\int_0^L \cos \frac{n\pi x}{L}\, dx = 0;$$

$$\int_{-L}^L \sin \frac{n\pi x}{L}\, dx = 0;$$

$$\int_{-L}^L \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx = 0;$$

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = \begin{cases} 0, & n \neq m, \\ L/2, & n = m; \end{cases}$$

$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx = \begin{cases} 0, & n \neq m, \\ L/2, & n = m; \end{cases}$$

$$\int_0^L \sin \frac{(2n-1)\pi x}{2L} \sin \frac{(2m-1)\pi x}{2L}\, dx = \begin{cases} 0, & n \neq m, \\ L/2, & n = m; \end{cases}$$

$$\int_0^L \cos \frac{(2n-1)\pi x}{2L} \cos \frac{(2m-1)\pi x}{2L}\, dx = \begin{cases} 0, & n \neq m, \\ L/2, & n = m. \end{cases}$$

General Regular Sturm-Liouville Problem

$$[p(x)f'(x)]' + q(x)f(x) + \lambda\sigma(x)f(x) = 0, \quad a < x < b, \quad p, \sigma > 0,$$
$$\kappa_1 f(a) + \kappa_2 f'(a) = 0, \quad \kappa_3 f(b) + \kappa_4 f'(b) = 0$$

(κ_1, κ_2 not both zero, κ_3, κ_4 not both zero).

Some Standard Eigenvalue-Eigenfunction Pairs

(i) If

$$f''(x) + \lambda f(x) = 0, \quad 0 < x < L, \quad f(0) = 0, \quad f(L) = 0,$$

then

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad f_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots.$$

(ii) If

$$f''(x) + \lambda f(x) = 0, \quad 0 < x < L, \quad f'(0) = 0, \quad f'(L) = 0,$$

then

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad f_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots.$$

(iii) If

$$f''(x) + \lambda f(x) = 0, \quad 0 < x < L, \quad f(0) = 0, \quad f'(L) = 0,$$

then

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, \quad f_n(x) = \sin\frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, \ldots.$$

(iv) If

$$f''(x) + \lambda f(x) = 0, \quad 0 < x < L, \quad f'(0) = 0, \quad f(L) = 0,$$

then

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, \quad f_n(x) = \cos\frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, \ldots.$$

(v) If

$$f''(x) + af'(x) + bf(x) + \lambda c f(x) = 0, \quad 0 < x < L, \quad f(0) = 0, \quad f(L) = 0,$$

then

$$\lambda_n = \frac{1}{4c}\left(n^2\pi^2 + a^2 - 4b\right), \quad f_n(x) = e^{-(a/2)x}\sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots.$$

Table of Full Fourier Transforms

$f(x) = \mathcal{F}^{-1}[F](x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega$	$F(\omega) = \mathcal{F}[f](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx$
$f'(x)$	$-i\omega F(\omega)$
$f''(x)$	$-\omega^2 F(\omega)$
$f(x-a)$	$e^{-ia\omega}$
e^{-ax^2}	$\frac{1}{\sqrt{4\pi a}}\, e^{-\omega^2/(4a)}$
$\frac{2a}{x^2+a^2}$	$e^{-a\|\omega\|}$
$\begin{cases} 1, & \|x\| < a \\ 0, & \|x\| > a \end{cases}$	$\frac{\sin(a\omega)}{\pi\omega}$
$(f * g)(x)$	$F(\omega)G(\omega)$

Table of Fourier Sine Transforms

$f(x) = \mathcal{F}_S^{-1}[F](x) = \int_0^\infty F(\omega)\sin(\omega x)\,d\omega$	$F(\omega) = \mathcal{F}_S[f](\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx$
$f'(x)$	$-\omega\mathcal{F}_C[f](\omega)$
$f''(x)$	$\frac{2\omega}{\pi}\,f(0) - \omega^2 F(\omega)$
1	$\frac{2}{\pi\omega}$
e^{-ax}	$\frac{2\omega}{\pi(a^2+\omega^2)}$
$\frac{x}{x^2+a^2}$	$e^{-a\omega}$
$\operatorname{erfc}(ax)$	$\frac{2}{\pi\omega}\left[1 - e^{-\omega^2/(4a^2)}\right]$

Table of Fourier Cosine Transforms

$f(x) = \mathcal{F}_C^{-1}[F](x) = \int_0^\infty F(\omega)\cos(\omega x)\,d\omega$	$F(\omega) = \mathcal{F}_C[f](\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)\,dx$
$f'(x)$	$-\frac{2}{\pi}\,f(0) + \omega\mathcal{F}_S[f](\omega)$
$f''(x)$	$-\frac{2}{\pi}\,f'(0) - \omega^2 F(\omega)$
e^{-ax}	$\frac{2a}{\pi(a^2+\omega^2)}$
e^{-ax^2}	$\frac{1}{\sqrt{\pi a}}\,e^{-\omega^2/(4a)}$
$\frac{a}{x^2+a^2}$	$e^{-a\omega}$

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}[F](t) = \dfrac{1}{2\pi i} \displaystyle\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds$	$F(s) = \mathcal{L}[f](s) = \displaystyle\int_0^{\infty} f(t)e^{-st}\,dt$
1	$\dfrac{1}{s} \qquad (s > 0)$
$t^n \qquad (n \text{ positive integer})$	$\dfrac{n!}{s^{n+1}} \qquad (s > 0)$
e^{at}	$\dfrac{1}{s-a} \qquad (s > a)$
$\sin(at)$	$\dfrac{a}{s^2+a^2} \qquad (s > 0)$
$\cos(at)$	$\dfrac{s}{s^2+a^2} \qquad (s > 0)$
$\sinh(at)$	$\dfrac{a}{s^2-a^2} \qquad (s > \lvert a \rvert)$
$\cosh(at)$	$\dfrac{s}{s^2-a^2} \qquad (s > \lvert a \rvert)$
$\delta(t-a)$	$e^{-as} \qquad (a \geq 0)$
$H(t-a)f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$(f * g)(t)$	$F(s)G(s)$
$f^{(n)}(t) \qquad (n\text{th derivative})$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$\operatorname{erf}\left(\dfrac{t}{2a}\right)$	$\dfrac{1}{s}\, e^{a^2 s^2} \operatorname{erfc}(as)$
$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{1}{s}\, e^{-a\sqrt{s}}$
$\dfrac{1}{\sqrt{\pi}}\, t^{-1/2}\, e^{-a^2/(4t)}$	$\dfrac{1}{\sqrt{s}}\, e^{-a\sqrt{s}} \qquad (a \geq 0)$
$\dfrac{a}{2\sqrt{\pi}}\, t^{-3/2}\, e^{-a^2/(4t)}$	$e^{-a\sqrt{s}} \qquad (a > 0)$

Green's Functions and Representation Formulas

(i) The solution of the IBVP

$$\begin{aligned} &u_t(x,t) = ku_{xx}(x,t) + q(x,t), \quad 0 < x < L,\ t > 0, \\ &u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad 0 < x < L, \end{aligned}$$

is given by the formula

$$u(x,t) = \int_0^L G(x,t;\xi,0)f(\xi)\,d\xi + \int_0^L \int_0^t G(x,t;\xi,\tau)q(\xi,\tau)\,d\tau\,d\xi,$$

where

$$G(x,t;\xi,\tau) = \sum_{n=1}^{\infty} \frac{2}{L} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L}\, e^{-k(n\pi/L)^2(t-\tau)}, \quad \tau < t.$$

(ii) The solution of the BVP

$$\begin{aligned} &(\Delta u)(x,y) = q(x,y), \quad 0 < x < L,\ 0 < y < K, \\ &u(x,0) = 0, \quad u(x,K) = 0, \quad 0 < x < L, \\ &u(0,y) = 0, \quad u(L,y) = 0, \quad 0 < y < K, \end{aligned}$$

is given by the formula

$$u(x,y) = \int_0^K \int_0^L G(x,y;\xi,\eta)q(\xi,\eta)\,d\xi\,d\eta,$$

where

$$G(x,y;\xi,\eta) = -\frac{4}{LK} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\sin(n\pi\xi/L)\sin(m\pi\eta/K)}{(n\pi/L)^2 + (m\pi/K)^2} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{K}.$$

(iii) The solution of the IVP

$$\begin{aligned} &u_{tt}(x,t) = c^2 u_{xx}(x,t) + q(x,t), \quad -\infty < x < \infty,\ t > 0, \\ &u(x,t),\ u_x(x,t) \to 0 \quad \text{as } x \to \pm\infty,\ t > 0, \\ &u(x,0) = 0, \quad u_t(x,0) = 0, \quad -\infty < x < \infty, \end{aligned}$$

is given by the formula

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,t;\xi,\tau)q(\xi,\tau)\,d\xi\,d\tau,$$

where

$$G(x,t;\xi,\tau) = \frac{1}{2c}\left[H\big(x - \xi + c(t-\tau)\big) - H\big(x - \xi - c(t-\tau)\big)\right].$$

Second-Order Linear Equations

If

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

if new variables

$$r = r(x,y), \quad s = s(x,y)$$

are defined by the characteristic equations

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}, \qquad \frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A},$$

and if

$$u(x,y) = u\big(x(r,s), y(r,s)\big) = v(r,s),$$

then

$$\bar{A}v_{rr} + \bar{B}v_{rs} + \bar{C}v_{ss} + \bar{D}v_r + \bar{E}v_s + \bar{F}v = \bar{G},$$

where

$$\begin{aligned}
\bar{A} &= A(r_x)^2 + Br_xr_y + C(r_y)^2, \\
\bar{B} &= 2Ar_xs_x + B(r_xs_y + r_ys_x) + 2Cr_ys_y, \\
\bar{C} &= A(s_x)^2 + Bs_xs_y + C(s_y)^2, \\
\bar{D} &= Ar_{xx} + Br_{xy} + Cr_{yy} + Dr_x + Er_y, \\
\bar{E} &= As_{xx} + Bs_{xy} + Cs_{yy} + Ds_x + Es_y, \\
\bar{F} &= F, \\
\bar{G} &= G,
\end{aligned}$$

with $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$, $\bar{F}$, and $\bar{G}$ expressed in terms of r and s.